Wilson–'t Hooft lines as transfer matrices

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Based on joint work with Kazunobu Maruyoshi and Toshihiro Ota

Various connections between

$$\text{supersymmetric QFTs} \longleftrightarrow \text{quantum integrable systems}$$

discovered in the past 10 years or so:

- Bethe/gauge correspondence (2d & 4d) [Nekrasov–Shatashvili]
- Bazhanov–Sergeev model from 4d $\mathcal{N} = 1$ quiver gauge theories [Spiridonov, Yamazaki]
- Surface defects as transfer matrices [Maruyoshi–Yagi]
- 4d Chern–Simons (= Ω-deformed 6d SYM [Costello–Y]) [Costello, Costello–Yamazaki–Witten]
- …

Many of them are related by string dualities [Costello–Y].

Quantization of Donagi–Witten integrable system

- $\mathcal{N} = 2$ theory on $\mathbb{R}^3 \times S^1$ on Coulomb branch
- IR: $\mathcal{N} = 4$ sigma model on $\mathbb{R}^3$
- Target $\mathcal{M}$ is the phase space of a classical complex integrable system [Donagi–Witten]
- Ω-deformation on $\mathbb{R}^2 \subset \mathbb{R}^3$ quantizes $\mathcal{M}$ [Nekrasov–Shatashvili, Nekrasov–Witten, Y]
- For class-$\mathcal{S}$ theories, $\mathcal{M}$ is a Hitchin system.

Surface defects as transfer matrices [Maruyoshi–Y, Y]

- $\mathcal{N} = 1$ theory constructed by "brane tiling" or of class $\mathcal{S}_k$
- Place it on $S^3 \times S^1$
- Insert surface defects on $S^1 \times S^1$
- Surface defects act on SUSY index as difference operators, shifting flavor fugacities [Gadde–Gukov, Gaiotto–Rastelli–Razamat]
- Coincide with transfer matrices of elliptic QIS [Maruyoshi–Y, Y]
- Simplest case: elliptic Ruijsenaars–Schneider system [GRR, Bullimore–Fluder–Hollands–Richmond]

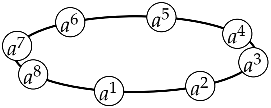
We found a new correspondence:

Wilson–'t Hooft lines = transfer matrices

- $\mathcal{N} = 2$ circular quiver theory (class-$\mathcal{S}$)
- Place it on $S^1 \times \mathbb{R}^3$
- Wind a Wilson–'t Hooft line T around S^1
- $\langle T \rangle$ is a function of Coulomb branch parameters
- Quantization of $\langle T \rangle$ coincides with transfer matrix of trigonometric QIS

Related to other correspondences

Consider a periodic spin chain



Spins $a^1, \ldots, a^n \in \mathfrak{h}^*$, $\mathfrak{h}$ = Cartan of $\mathfrak{sl}_N$:

$$a^r = \mathrm{diag}(a^r_1, \ldots, a^r_N)\,, \qquad \sum_{i=1}^{N} a^r_i = 0$$

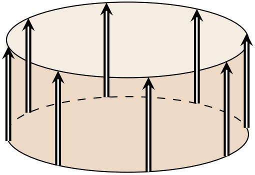
Local Hilbert space:

$$\mathcal{M}_{\mathfrak{h}^*} = \{\text{meromorphic functions on } \mathfrak{h}^*\}$$

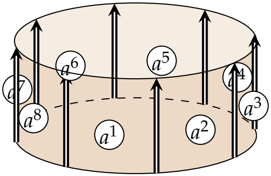
Total Hilbert space

$$\mathcal{H} = \underbrace{\mathcal{M}_{\mathfrak{h}^*} \otimes \cdots \otimes \mathcal{M}_{\mathfrak{h}^*}}_{n}$$

Equivalent lattice model

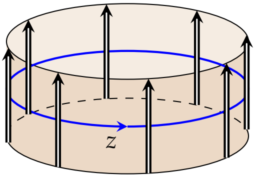


Spins live between double lines:



a^r are called dynamical parameters.

Transfer matrix $T(z)$ is horizontal loop operator:

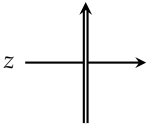


Solid line = worldline of particle whose Hilbert space is $\mathbb{C}^N$

The particle's state changes when it crosses other lines.

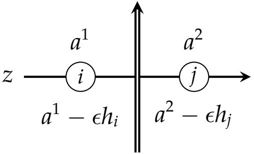
Solid line also has spectral parameter $z \in \mathbb{C}$.

$T(z)$ consists of n copies of L-operator

$L(z) =$  .

Dynamical parameters jump across solid lines:

$$L(z;a^1,a^2)_i^j =$$



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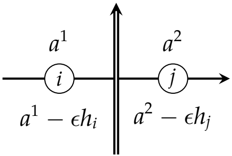
$\epsilon \in \mathbb{C}$: fixed parameter (Planck constant)

h_i are the weights of the vector rep $\mathbb{C}^N$:

$$\begin{aligned}
h_1 &= \mathrm{diag}(1-\tfrac{1}{N}, -\tfrac{1}{N}, -\tfrac{1}{N}, \dots, -\tfrac{1}{N}), \\
h_2 &= \mathrm{diag}(-\tfrac{1}{N}, 1-\tfrac{1}{N}, -\tfrac{1}{N}, \dots, -\tfrac{1}{N}), \\
&\vdots \\
h_N &= \mathrm{diag}(-\tfrac{1}{N}, -\tfrac{1}{N}, -\tfrac{1}{N}, \dots, 1-\tfrac{1}{N}).
\end{aligned}$$

$$L(z; a^1, a^2)_i^j = \; z$$



Matrix elements $L(z)_i^j$ are difference operators on $\mathcal{M}_{\mathfrak{h}^*} \otimes \mathcal{M}_{\mathfrak{h}^*}$:

$$L(z) = \sum_{i,j} L(z; a^1, a^2)_i^j \Delta_i^1 \Delta_j^2 \,,$$

$$\Delta_i^r : a^r \mapsto a^r - \epsilon h_i \,.$$

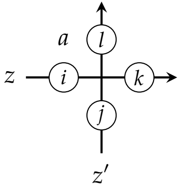
Transfer matrix

$$T(z) = \sum_{i^1, \ldots, i^n} \prod_{r=1}^{n} L(z; a^r, a^{r+1})_{i^r}^{i^{r+1}} \prod_{s=1}^{n} \Delta_{i^s}^s \,, \qquad i^{n+1} = i^1$$

is a difference operator on $\mathcal{H} = \mathcal{M}_{\mathfrak{h}^*}^{\otimes n}$.

Crossing solid lines give R-matrix

$$R(z - z'; a)_{ij}^{kl} =$$



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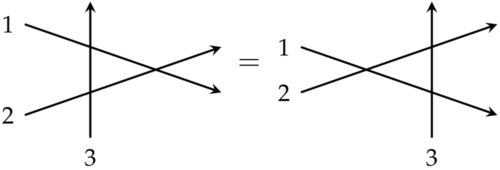
R-matrix satisfies dynamical Yang–Baxter equation



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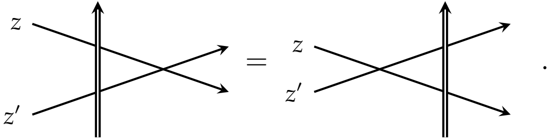
Just like the ordinary Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

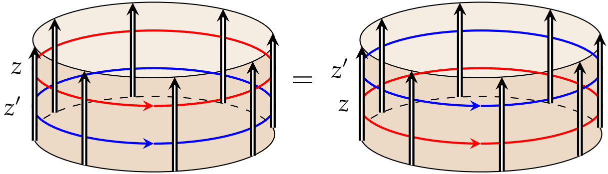
but with shifts in the dynamical parameters.

L-operator and R-matrix satisfy RLL relation



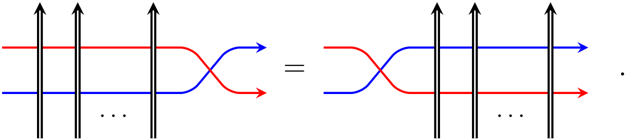
It follows that transfer matrices commute:

$$T(z)T(z') = T(z')T(z)$$

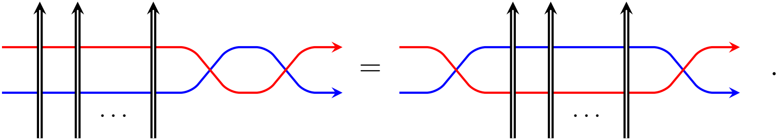


Proof:

By RLL relation



Multiply both sides by R^{-1}:



Take the trace, making the horizontal direction periodic.

Since

$$[T(z), T(z')] = 0\,,$$

coefficients of Laurent expansion

$$T(z) = \sum_{m=-\infty}^{\infty} T_m z^m$$

are commuting difference operators on $\mathcal{H}$:

$$[T_m, T_n] = 0\,.$$

This is integrability.

Trigonometric L-operator [Hasegawa]

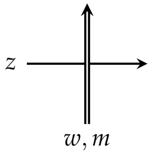
$$\mathcal{L}_{w,m}(z)_i^j = \sum_{i,j} (\Delta_i^1 \Delta_j^2)^{\frac{1}{2}} \frac{\sin \pi (z - w + a_j^2 - a_i^1)}{\sin \pi (z - w)} \ell_m(a^1, a^2)_i^j (\Delta_i^1 \Delta_j^2)^{\frac{1}{2}}$$

satisfies RLL relation with a trigonometric dynamical R-matrix (a limit of the 8vSOS R-matrix).

$$\ell_m(a^1, a^2)_i^j = \left(\frac{\prod_{k(\neq i)} \sin \pi (a_k^1 - a_j^2 - m) \prod_{l(\neq j)} \sin \pi (a_i^1 - a_l^2 - m)}{\prod_{k(\neq i)} \sin \pi (a_{ki}^1 - \frac{1}{2}\epsilon) \sin \pi (a_{ik}^1 - \frac{1}{2}\epsilon)} \right)^{\frac{1}{2}}$$

w, $m \in \mathbb{C}$ are spectral parameters assigned to the double line:

$\mathcal{L}_{w,m}(z) =$



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Introduce fundamental L-operators

$$\mathcal{L}_{\pm,m} = \lim_{w \to \pm \mathrm{i}\infty} \mathcal{L}_{w,m} \,.$$

Then

$$(\mathcal{L}_{\pm,m})_i^j = \sum_{i,j} (\Delta_i^1 \Delta_j^2)^{\frac{1}{2}} e^{\pm \pi \mathrm{i}(a_j^2 - a_i^1)} \ell_m(a^1, a^2)_i^j (\Delta_i^1 \Delta_j^2)^{\frac{1}{2}}$$

and

$$\mathcal{L}_{w,m}(z) = \frac{e^{\pi \mathrm{i}(z-w)} \mathcal{L}_{+,m} - e^{-\pi \mathrm{i}(z-w)} \mathcal{L}_{-,m}}{\sin \pi (z-w)} \,.$$

We may as well consider $\mathcal{L}_{\pm,m}$ without loss of generality.

Pick n-tuple of signs

$$\sigma = (\sigma^1, \ldots, \sigma^n) \in \{\pm\}^n$$

and n-tuple of complex numbers

$$m = (m^1, \ldots, m^n) \in \mathbb{C}^n .$$

Let $\mathcal{T}_{\sigma,m}$ be the transfer matrix constructed from n L-operators

$$\mathcal{L}_{\sigma^1, m^1} \, , \ \ldots \ , \mathcal{L}_{\sigma^n, m^n} \, .$$

$$\mathcal{T}_{\sigma,m} = \sum_{i^1,\ldots,i^n} \left(\prod_{s=1}^{n} \Delta^s_{i_s} \right)^{\frac{1}{2}} \prod_{r=1}^{n} e^{\pi \mathrm{i} \sigma^r (a^{r+1}_{i^{r+1}} - a^r_{i^r})} \ell_{m^r}(a^r, a^{r+1})^{i^{r+1}}_{i^r} \left(\prod_{s=1}^{n} \Delta^s_{i_s} \right)^{\frac{1}{2}} .$$

This is the main character on the integrable system side.

$\mathcal{N} = 2$ gauge theories have half-BPS Wilson–'t Hooft lines.

Wolrdlines of very massive dyonic particles

Charge of WH line

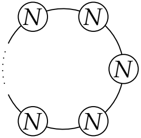
$$(\mathbf{m}, \mathbf{e}) \in (\Lambda_{\text{coweight}} \times \Lambda_{\text{weight}})/\text{Weyl}\,.$$

Wilson line has $\mathbf{m} = 0$ and is labeled by representation of $\mathfrak{g}$.

't Hooft line has $\mathbf{e} = 0$ and is labeled by representation of ${}^L\mathfrak{g}$.

Wilson–'t Hooft
= ('t Hooft) + (Wilson for subgroup of G leaving $\mathbf{m}$ invariant)

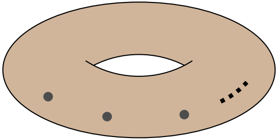
$\mathcal{N} = 2$ gauge theory described by n-node circular quiver



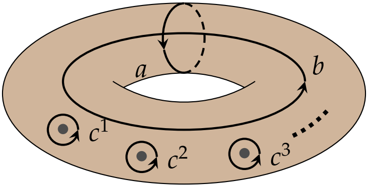
Each node is $\mathrm{SU}(N)$ (more precisely, $\mathrm{PSU}(N)$).

Edges are bifundamental hypers with masses $m^1, \ldots, m^n$.

Compactification of 6d $\mathcal{N} = (2,0)$ SCFT on n-punctured torus



WH lines = surface defects wrapping 1-cycles of the torus



Consider Wilson–'t Hooft line $T_{\square,\sigma}$ corresponding to

$$\gamma_\sigma = b + \sum_r \frac{1-\sigma^r}{2} c^r \, .$$

If $\sigma^r = +1$ (-1), the cycle passes above (below) rth puncture.

$\mathbf{m} = \square \oplus \cdots \oplus \square$ under $\mathfrak{su}_N \oplus \cdots \oplus \mathfrak{su}_N$

$\mathbf{e}$ specified by $\sigma \in \{\pm\}^n$

Put the theory on twisted product

$$S^1 \times_\epsilon \mathbb{R}^2 \times \mathbb{R}\,.$$

Wrap $T_{\square,\sigma}$ around $S^1 \times \{0\} \times \{t\}$.

Ito–Okuda–Taki tell us how to compute the vev by localization:

$$\langle T_{\square,\sigma} \rangle = \sum_{i^1,\dots,i^n} \prod_{r=1}^{n} e^{2\pi \mathrm{i} b^r_{i^r}} e^{\pi \mathrm{i} \sigma^r (a^{r+1}_{i^{r+1}} - a^r_{i^r})} \ell_{m^r}(a^r, a^{r+1})^{i^{r+1}}_{i^r}$$

in complexified Fenchel–Nielsen coordinates on Seiberg–Witten moduli space:

$$a = \frac{\theta_{\mathrm{e}}}{2\pi} + \mathrm{i}\beta \operatorname{Re}\phi + \cdots\,, \quad b = \frac{\theta_{\mathrm{m}}}{2\pi} - \frac{4\pi \mathrm{i}\beta}{g^2} \operatorname{Im}\phi + \mathrm{i}\frac{\vartheta}{2\pi}\beta \operatorname{Re}\phi + \cdots\,.$$

Alternatively, we can compute it from Toda theory by AGT.

Compare

$$\langle T_{\square,\sigma} \rangle = \sum_{i^1,\dots,i^n} \prod_{r=1}^{n} e^{2\pi \mathrm{i} b^r_{i^r}} e^{\pi \mathrm{i} \sigma^r (a^{r+1}_{i^{r+1}} - a^r_{i^r})} \ell_{m^r}(a^r, a^{r+1})^{i^{r+1}}_{i^r} \,,$$

$$\mathcal{T}_{\sigma,m} = \sum_{i^1,\dots,i^n} \left(\prod_{s=1}^{n} \Delta^s_{i_s} \right)^{\frac{1}{2}} \prod_{r=1}^{n} e^{\pi \mathrm{i} \sigma^r (a^{r+1}_{i^{r+1}} - a^r_{i^r})} \ell_{m^r}(a^r, a^{r+1})^{i^{r+1}}_{i^r} \left(\prod_{s=1}^{n} \Delta^s_{i_s} \right)^{\frac{1}{2}} .$$

If we quantize a^r, b^r so that

$$[\hat{a}^r_i, \hat{b}^s_j] = -\mathrm{i} \frac{\epsilon}{2\pi} \delta^{rs} \left(\delta_{ij} - \frac{1}{N} \right),$$

then

$$\mathcal{T}_{\sigma,m} = \text{Weyl quantization of } \langle T_{\square,\sigma} \rangle \,.$$

LHS easily generalizes to other reps by fusion procedure, RHS does not due to monopole bubbling.

M-theory setup

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\mathbb{R}_9$	S^1_{10}
N M5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	S^1_6	—	—	—	S^1_{10}
n M5′	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	—	—	$\mathbb{R}_8$	$\mathbb{R}_9$	—
M2	—	—	S^1_3	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—	—

12345 directions: twisted product $\mathbb{R}^2_{12} \times_\epsilon S^1_3 \times_{-\epsilon} \mathbb{R}^2_{45}$

M5: 6d $\mathcal{N} = (2,0)$ SCFT on $\mathbb{R}_0 \times \mathbb{R}^2_{12} \times_\epsilon S^1_3 \times S^1_6 \times S^1_{10}$

M5′: n punctures on $S^1_6 \times S^1_{10}$

M2: surface defect

Reduction on $S^1_6 \times S^1_{10}$ gives the 4d setup with $\sigma = (+, \ldots, +)$.

Compactify $\mathbb{R}_9 \to S^1_9$:

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	S^1_9	S^1_{10}
N M5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	S^1_6	—	—	—	S^1_{10}
n M5′	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	—	—	$\mathbb{R}_8$	S^1_9	—
M2	—	—	S^1_3	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—	—

Reduce on S^1_3:

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	S^1_9	S^1_{10}
N D4	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	S^1_6	—	—	—	S^1_{10}
n D4	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	—	—	$\mathbb{R}_8$	S^1_9	
F1	—	—	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—	—

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	S^1_9	S^1_{10}
N D4	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	S^1_6	—	—	—	S^1_{10}
n D4	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	—	—	$\mathbb{R}_8$	S^1_9	
F1	—	—	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—	—

Apply T-duality $S^1_9 \to \check{S}^1_9$:

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$	S^1_{10}
N D5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	S^1_6	—	—	$\check{S}^1_9$	S^1_{10}
n D3	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	—	—	$\mathbb{R}_8$	—	
F1	—	—	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—	—

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$	S^1_{10}
N D5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	S^1_6	—	—	$\check{S}^1_9$	S^1_{10}
n D3	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	—	—	$\mathbb{R}_8$	—	
F1	—	—	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—	—

D5: 6d $\mathcal{N} = (1,1)$ SYM on $\mathbb{R}_0 \times \mathbb{R}^2_{12} \times S^1_6 \times \check{S}^1_9 \times S^1_{10}$

D3: codim-3 operator on $\mathbb{R}_0 \times \mathbb{R}^2_{12}$

F1: Wilson line on S^1_6

Ω-deformation on $\mathbb{R}^2_{12}$ from nontrivial background, due to the initial twisted product in 12345 directions [Hellerman–Orland–Reffert].

Ω-deformed 6d $\mathcal{N} = (1,1)$ SYM on $\mathbb{R}_0 \times \mathbb{R}^2_{12} \times S^1_6 \times \check{S}^1_9 \times S^1_{10}$

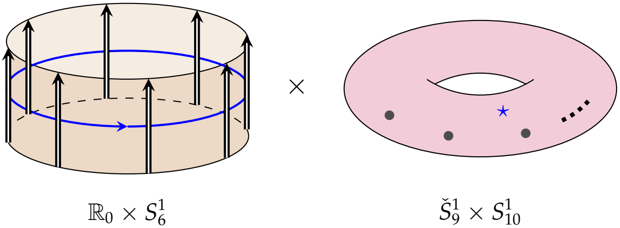
$\rightsquigarrow$ Costello's 4d Chern–Simons on $\mathbb{R}_0 \times S^1_6 \times \check{S}^1_9 \times S^1_{10}$ [Costello–Y]

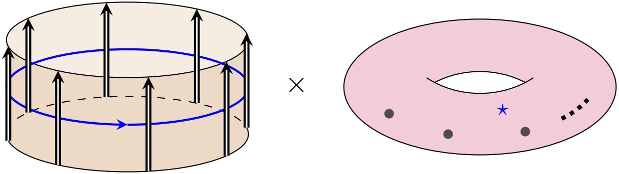
Codim-3 operators on $\mathbb{R}_0 \times \mathbb{R}^2_{12}$

$\rightsquigarrow$ line operators on $\mathbb{R}_0$

Wilson line on S^1_6

$\rightsquigarrow$ Wilson line on S^1_6



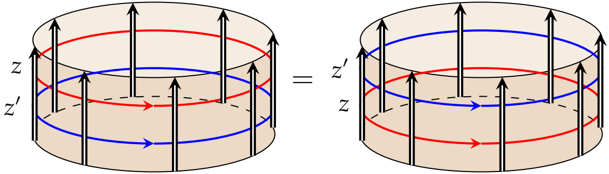


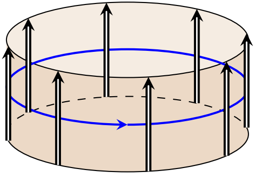
$\mathbb{R}_0 \times S^1_6$ $\check{S}^1_9 \times S^1_{10}$

Topological on $\mathbb{R}_0 \times S^1_6$, holomorphic on $\check{S}^1_9 \times S^1_{10}$

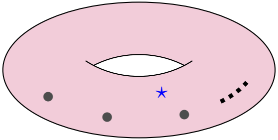
2d TQFT + line defects $\implies$ lattice model

TQFT + extra dimensions $\implies$ integrability [Costello]





$\mathbb{R}_0 \times S^1_6$



$\check{S}^1_9 \times S^1_{10}$

Wilson line gives transfer matrix of elliptic QIS with

$$\tau = \mathrm{i}R_{10}/\check{R}_9 \, .$$

Now, decompactify $S^1_9 \to \mathbb{R}_9$. Take $R_9 \to \infty$, or $\check{R}_9 \to 0$.

This is the trigonometric limit $\tau \to \mathrm{i}\infty$.

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$	S^1_{10}
N D5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	S^1_6	—	—	$\check{S}^1_9$	S^1_{10}
n D3	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	—	—	$\mathbb{R}_8$	—	—
F1	—	—	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—	—

For Nekrasov–Shatashvili, apply S-duality:

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$	S^1_{10}
N NS5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	S^1_6	—	—	$\check{S}^1_9$	S^1_{10}
n D3	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	—	—	$\mathbb{R}_8$	—	—
D1	—	—	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—	—

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$	S^1_{10}
N NS5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	S^1_6	—	—	$\check{S}^1_9$	S^1_{10}
n D3	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	—	—	$\mathbb{R}_8$	—	—
D1	—	—	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—	—

Then T-duality on S^1_6:

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$\mathbb{R}^2_{45}$	$\check{S}^1_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$	S^1_{10}
N NS5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	$\check{S}^1_6$	—	—	$\check{S}^1_9$	S^1_{10}
n D4	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	$\check{S}^1_6$	—	$\mathbb{R}_8$	—	—
D0	—	—	—	—	—	$\mathbb{R}^{\geq 0}_8$	—	—

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	$\mathbb{R}^2_{45}$	$\check{S}^1_6$	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$	S^1_{10}
N NS5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	$\check{S}^1_6$	—	—	$\check{S}^1_9$	S^1_{10}
n D4	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	—	$\check{S}^1_6$	—	$\mathbb{R}_8$	—	—
D0	—	—	—	—	—	$\mathbb{R}^{\geq 0}_8$	—	—

D4–NS5: 4d $\mathcal{N} = 2$ theory for $(N+1)$-node linear quiver

$$\boxed{n} - (n) - \cdots - (n) - \boxed{n}$$

placed on $\mathbb{R}_0 \times \mathbb{R}^2_{12} \times \check{S}^1_6$.

Ω-deformation quantizes DW system (trigonometric Gaudin)
$\Longrightarrow$ noncompact XXX spin chain

D0 is a local operator, acting as a transfer matrix.

Actuality, 9 & 10 directions are compact, so it's a 6d lift. We get the elliptic version of the integrable system.

Go back to M-theory

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	S^1_9	S^1_{10}
N M5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	S^1_6	—	—	—	S^1_{10}
n M5′	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	—	—	$\mathbb{R}_8$	S^1_9	—
M2	—	—	S^1_3	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—	—

Reduce on S^1_{10}:

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$
N D4	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	S^1_6	—	—	—
n NS5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	—	—	$\mathbb{R}_8$	$\check{S}^1_9$
D2	—	—	S^1_3	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$
N D4	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	S^1_6	—	—	—
n NS5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	—	—	$\mathbb{R}_8$	$\check{S}^1_9$
D2	—	—	S^1_3	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	—

Apply T-duality on S^1_9:

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$
N D5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	S^1_6	—	—	$\check{S}^1_9$
n NS5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	—	—	$\mathbb{R}_8$	$\check{S}^1_9$
D3	—	—	S^1_3	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	$\check{S}^1_9$

Spacetime	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	$\mathbb{R}^2_{45}$	S^1_6	$\mathbb{R}_7$	$\mathbb{R}_8$	$\check{S}^1_9$
N D5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	S^1_6	—	—	$\check{S}^1_9$
n NS5	$\mathbb{R}_0$	$\mathbb{R}^2_{12}$	S^1_3	—	—	—	$\mathbb{R}_8$	$\check{S}^1_9$
D3	—	—	S^1_3	—	S^1_6	—	$\mathbb{R}^{\geq 0}_8$	$\check{S}^1_9$

D5–NS5: 5d circular quiver theory on $\mathbb{R}_0 \times \mathbb{R}^2_{12} \times_\epsilon S^1_3 \times \check{S}^1_9$

D3: surface defect on $S^1_3 \times \check{S}^1_9$

We can add more NS5s, preserving 4d $\mathcal{N} = 1$ SUSY on $\mathbb{R}^2_{12} \times_\epsilon S^1_3 \times \check{S}^1_9$. This leads to the brane tiling story [Maruyoshi–Y].

Summary

- We considered a class of Wilson–'t Hooft lines in 4d $\mathcal{N} = 2$ circular quiver theories.
- We found that they can be identified with transfer matrices of trigonometric QIS.
- This is useful for calculuations of line operator vevs.
- By embedding into string theory, the correspondence can be related to other known correspondences via dualities.

Further directions

- Surface defects in 5d circular quiver theory correspond to transfer matrices of elliptic QIS.
- Variations of the present setup
- Circular quiver theories deconstruct 6d $\mathcal{N} = (2, 0)$ SCFT. Integrability is behind surface operators in 6d theory.