2D dilaton-gravity and matrix models

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Plan of the talk

- Part 1: Pure Jackiw-Teitelboim gravity [Saad Shenker Stanford 19]
- Part 2: 2D dilaton-gravity [Maxfield, GJT 20] [Witten 20]
- Part 3: Connection to the minimal string [Mertens, GJT 20] [Usatyuk, Weng, GJT 20]

Jackiw-Teitelboim Gravity

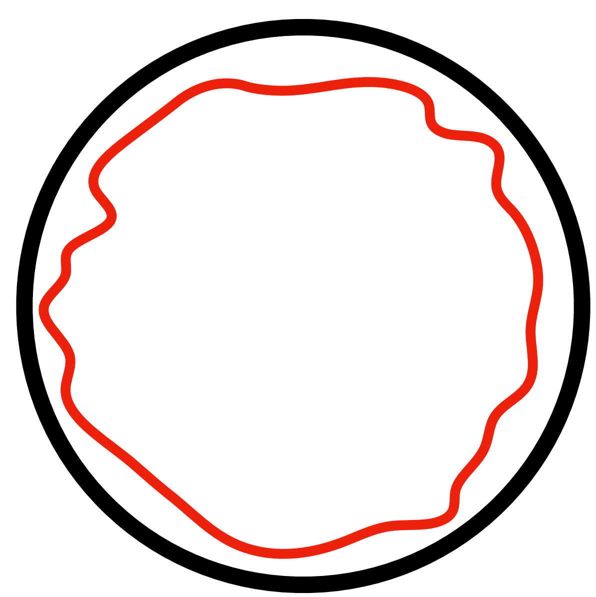
- Simple two dimensional theory of dilaton-gravity

$$I_{JT} = -\frac{S_0}{4\pi}\int R - \frac{1}{2}\int \phi(R+2) + I_{GHY}$$

- Asymptotically AdS_2 boundary conditions

$$\phi|_{\mathrm{bdy}} = \frac{\gamma}{\varepsilon}, \quad L|_{\mathrm{bdy}} = \frac{\beta}{\varepsilon}$$

$$\varepsilon \to 0$$



- Theory reduces to a boundary mode

- Broken conformal symmetry

[Almheiri, Polchinski 14] [Jensen 16]
[Maldacena,Stanford,Yang16] [Englesoy, Mertens Verlinde 16]…

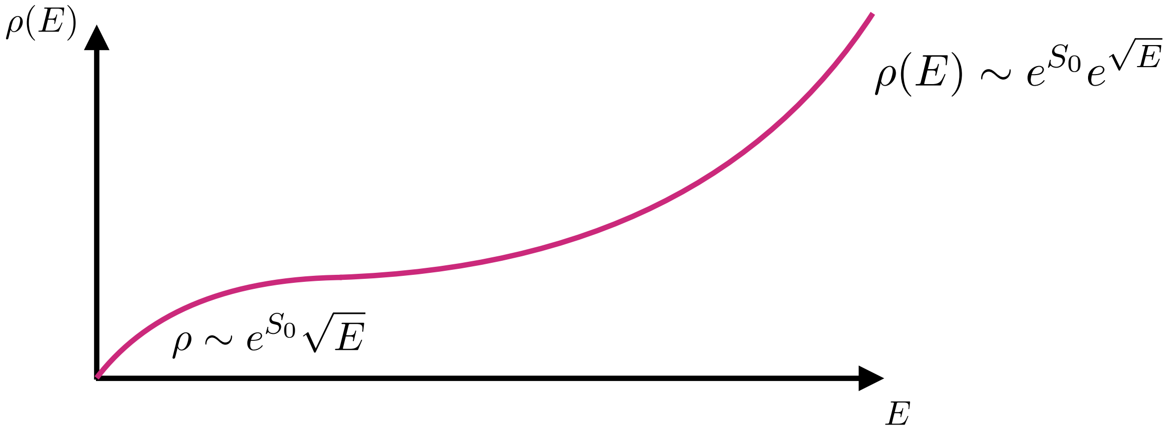
- Final answer for disk partition function

$$Z_{\text{disk}}(\beta) = e^{S_0}\sqrt{\frac{\gamma^3}{2\pi\beta^3}}\,e^{\frac{2\pi^2\gamma}{\beta}}$$

[Altland, Bagrets, Kamenev 16] [Stanford, Witten 17]
[Mertens, GJT, Verlinde 17]

- Density of states:

$$\rho_{\text{JT}}(E) = \frac{e^{S_0}}{4\pi^2}\sinh\left(2\pi\sqrt{2\gamma E}\right)$$



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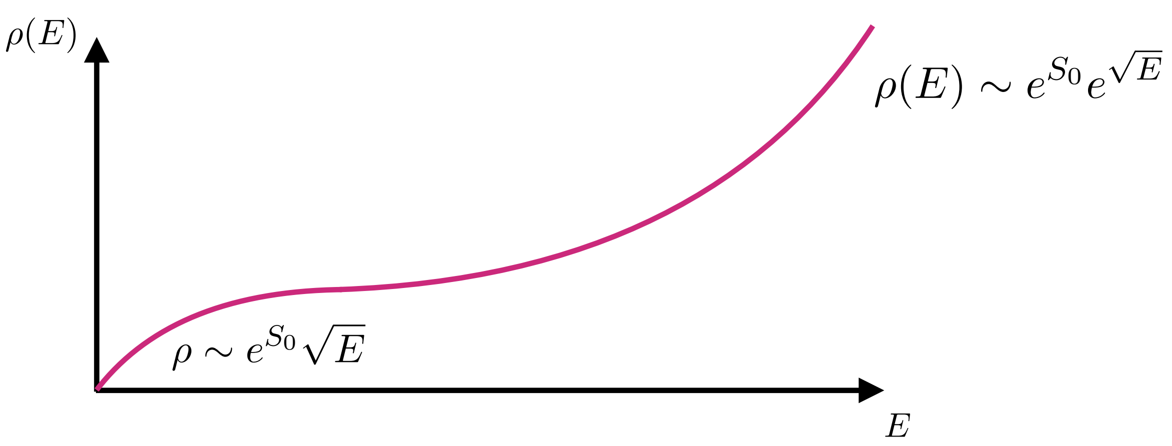
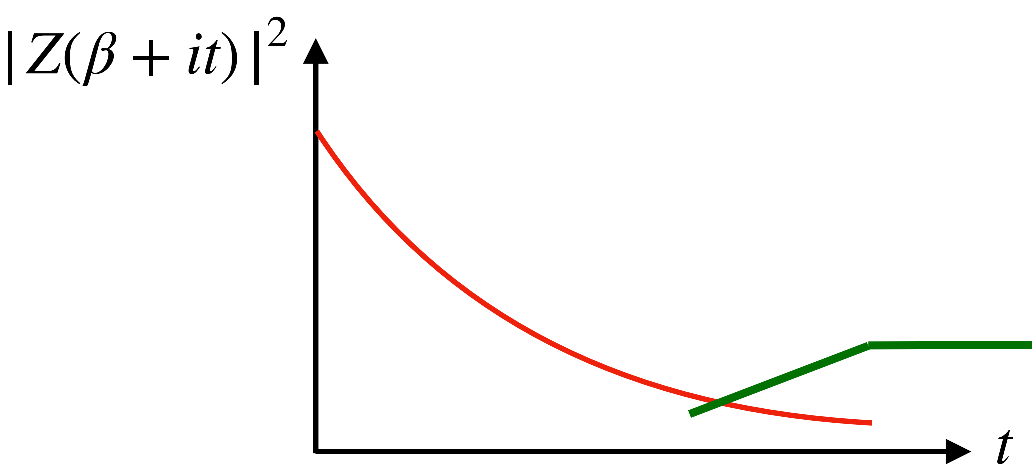


- Problem: The spectrum is continuous! SFF decays in time forever, matter correlators, etc…



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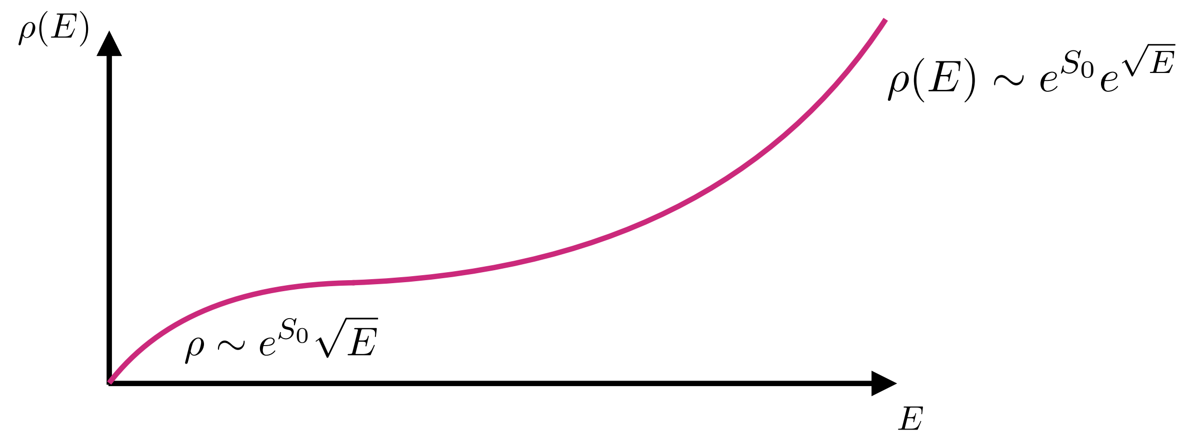


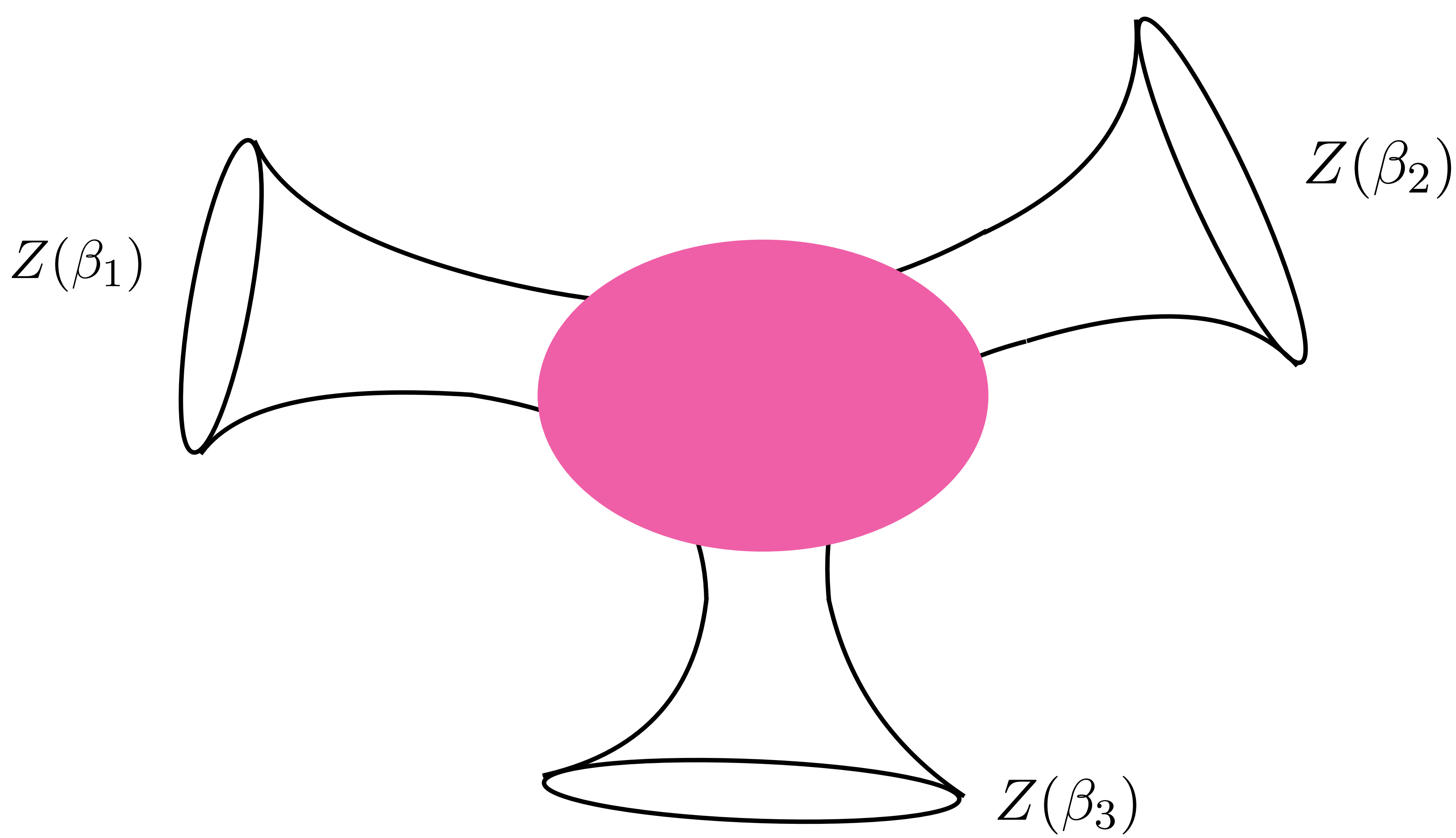
- Problem: The spectrum is continuous! SFF decays in time forever, matter correlators, etc…

- Saad-Shenker-Stanford: This can be solved by allowing to sum over topologies

Sum over topologies

- In 2D topologies are classified by genus. We will also include the possibility of having any number of boundaries

$$Z_{\text{grav}}(\beta_1, \ldots, \beta_n) = \int \mathcal{D}g\mathcal{D}\phi \; e^{-I_{\text{JT}}[g,\phi]}$$



Summary of Steps

Full gravitational path integral with n boundaries β_i and genus g

$$Z_{\mathrm{grav}}(\beta_1, \ldots, \beta_n) = \sum_{g=0}^{\infty} e^{-(2g-2+n)S_0} Z_{g,n}(\beta_1, \ldots, \beta_n)$$

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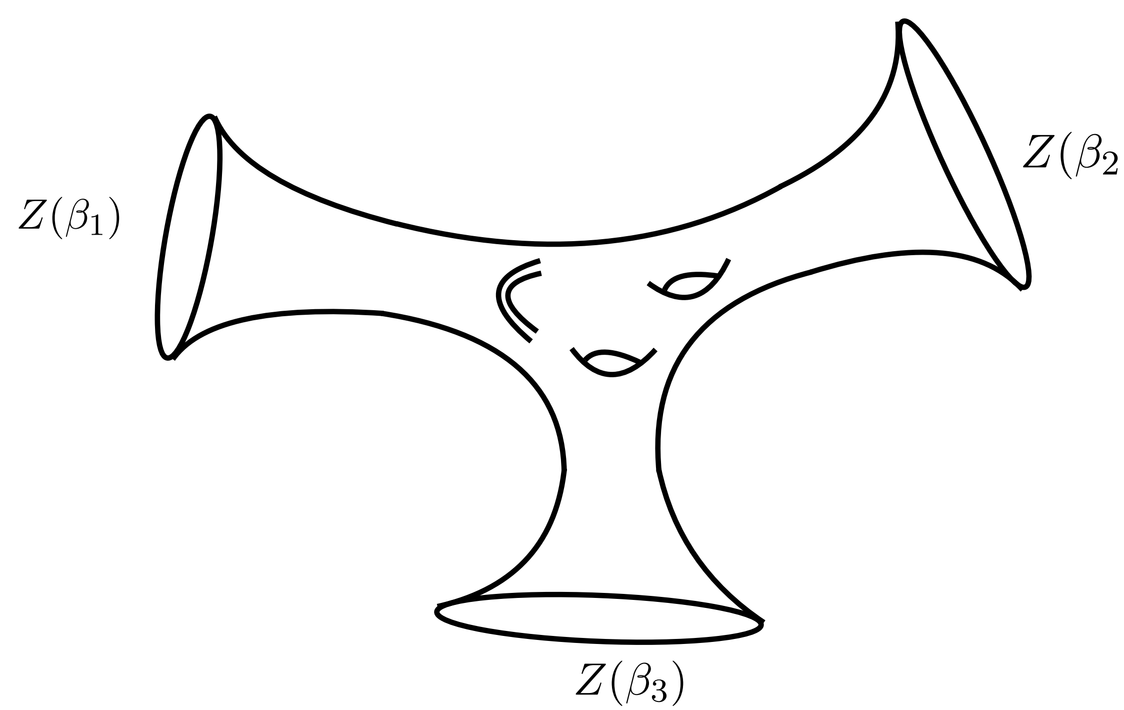
↓ Integrate out the dilaton

Integrate over hyperbolic surfaces with $R + 2 = 0$



Summary of Steps

Full gravitational path integral with n boundaries β_i and genus g

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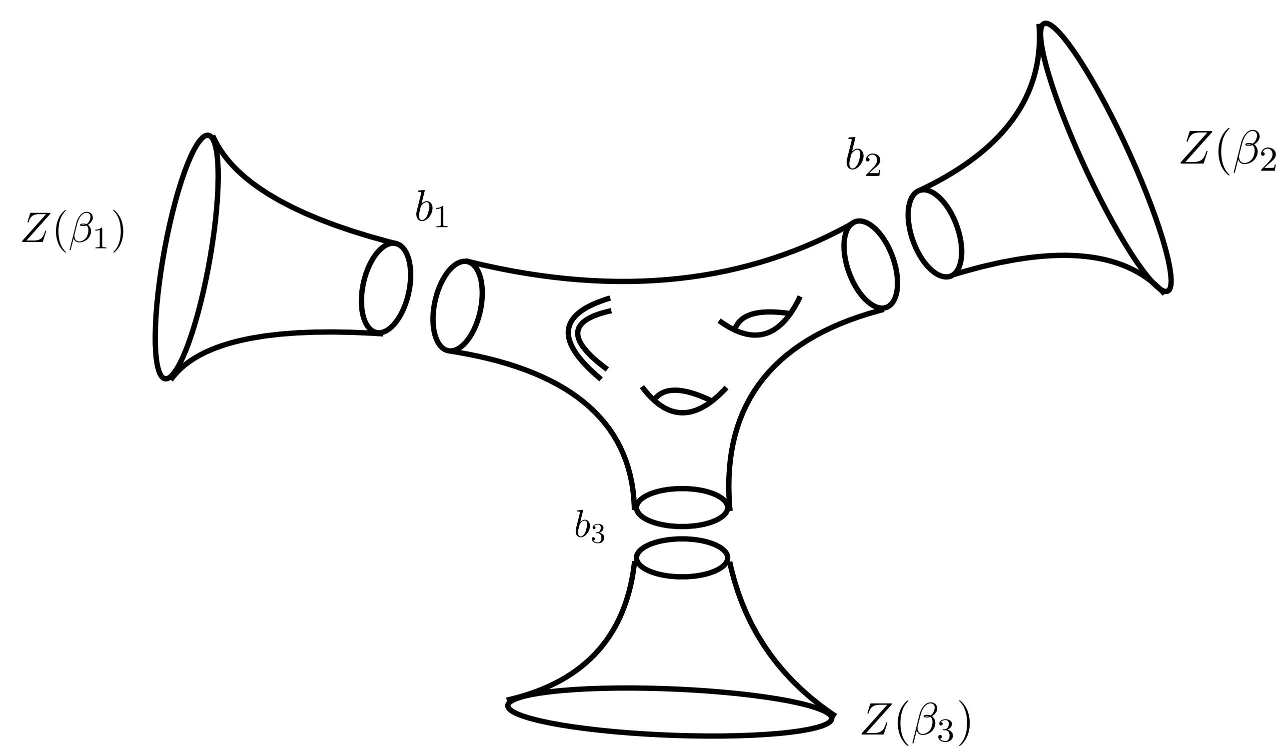
Integrate over hyperbolic surfaces with $R + 2 = 0$

Identify geodesics homologous to boundaries

Compute partition function of "trumpet" and bulk with geodesic boundaries



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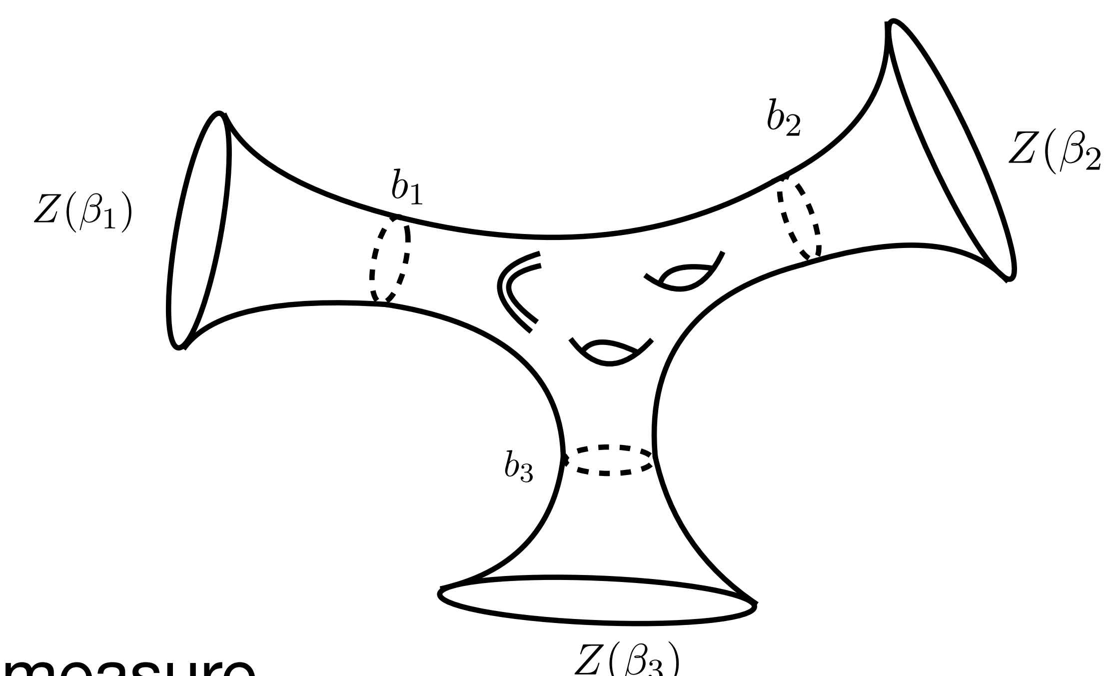
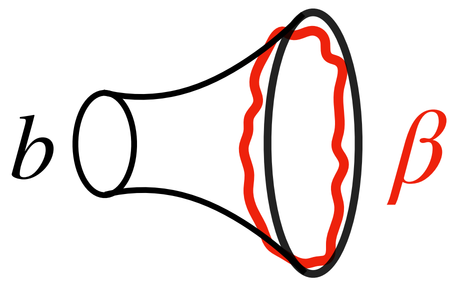


"Tree level exact": glue with Weil-Petersson measure

Ingredients

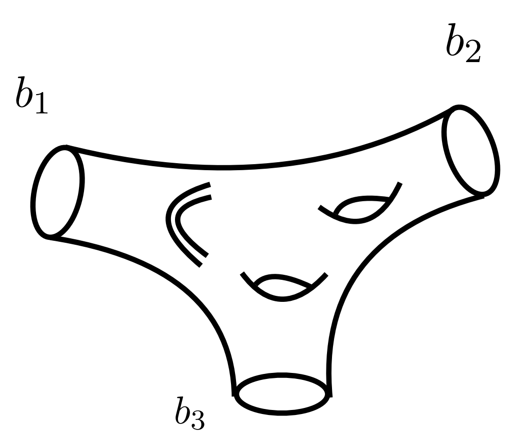
- Trumpet:



- Integrate out dilaton. "b" dependent boundary Schwarzian mode. Final answer:

$$Z_{\text{trumpet}}(\beta, b) = \sqrt{\frac{\gamma}{2\pi\beta}} e^{-\frac{\gamma}{2}\frac{b^2}{\beta}}$$

- Hyperbolic surfaces with geodesic boundaries:



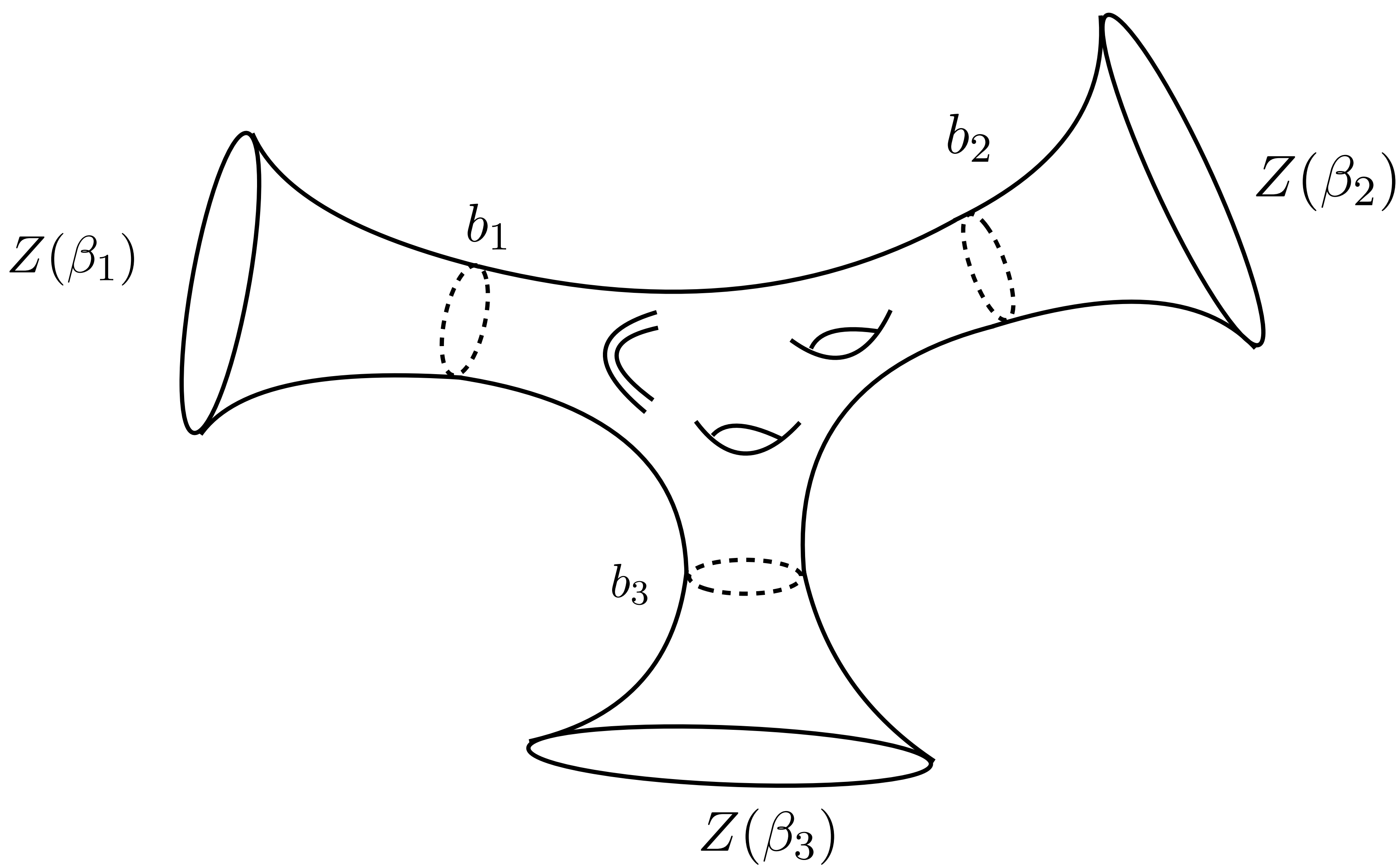
$$\rightarrow \qquad V_{g,n}(b_1, \ldots, b_n)$$

$\Rightarrow$ Weil-Petersson volumes, computed using Mirzakhani recursion

Sum over topologies

- Final answer obtained by gluing:

$$Z_{g,n}(\beta_1,\ldots,\beta_n) = \int_0^\infty b_1 db_1 Z_{\text{trumpet}}(\beta_1,b_1)\cdots\int_0^\infty b_n db_n Z_{\text{trumpet}}(\beta_n,b_n) V_{g,n}(b_1,\cdots,b_n)$$



JT gravity and matrices

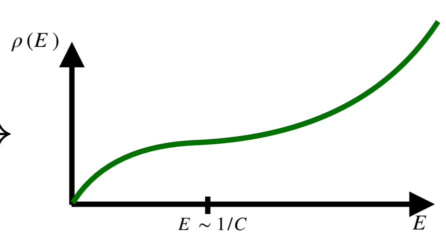
- SSS realized that the theory is equivalent, in a holographic sense, to a matrix integral of size $L \times L$, with $L \sim e^{S_0}$, such that

$$Z_{\text{grav}}(\beta_1, \dots, \beta_n) = \int dH\ P(H)\ \text{Tr}\left(e^{-\beta_1 H}\right) \dots \text{Tr}\left(e^{-\beta_n H}\right)$$

QM partition function

Probability distribution over Hamiltonians:

$$P(H) = e^{-L\text{Tr}\left(V(H)\right)} \quad \leftrightarrow$$



- Based on comparing Mirzakhani's recursion for Weil-Petersson volumes with matrix model loop equations

$$\langle Z(\beta_1)\dots Z(\beta_n)\rangle_{\text{conn.}} = \sum_{g=0}^{\infty} e^{-(2g-2+n)S_0} Z_{g,n}(\beta_1,\dots,\beta_n)$$

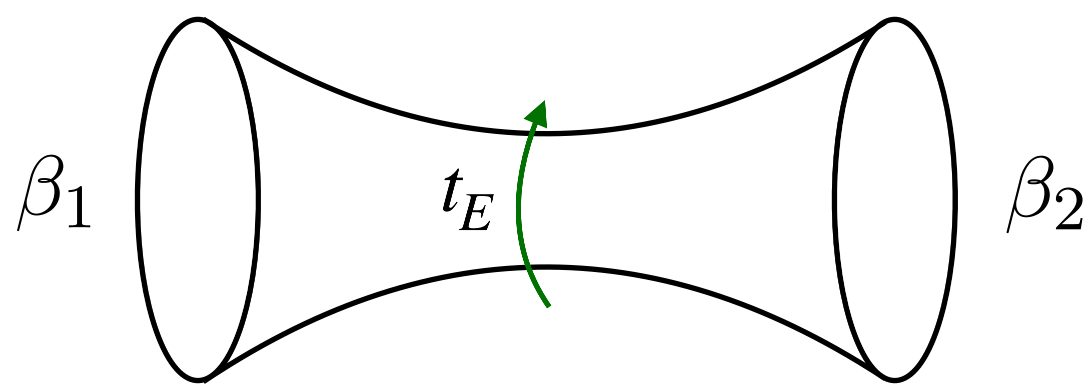
- **JT gravity:** Computed in terms of WP volumes. They satisfy a recursion of their own found by Mirzakhani.
- **Matrix Integral:** Computed in terms of the topological recursion of matrix models with $\rho_{\text{disk}} = \rho_{JT}(E)$

Eynard and Orantin proved that both recursions are identical (up to an integral transform)

SSS: This implies that pure JT gravity is holographically dual to a matrix integral, interpreted as an average over Hamiltonians

The Factorization Puzzle

- Smoking gun of holography with disorder: [Yau Witten 99] …



$$Z_{\text{grav}}(\beta_1, \beta_2) \neq Z_{\text{grav}}(\beta_1) Z_{\text{grav}}(\beta_2)$$

- A possible answer is to add “baby universe” Hilbert space.

[Coleman; Giddings Strominger 88]

- This does not quite work when computing entanglement entropies: an average quantum system is not necessarily a quantum system

[Giddings GJT 20]

JT gravity with a gas of defects

- Motivations for doing this:

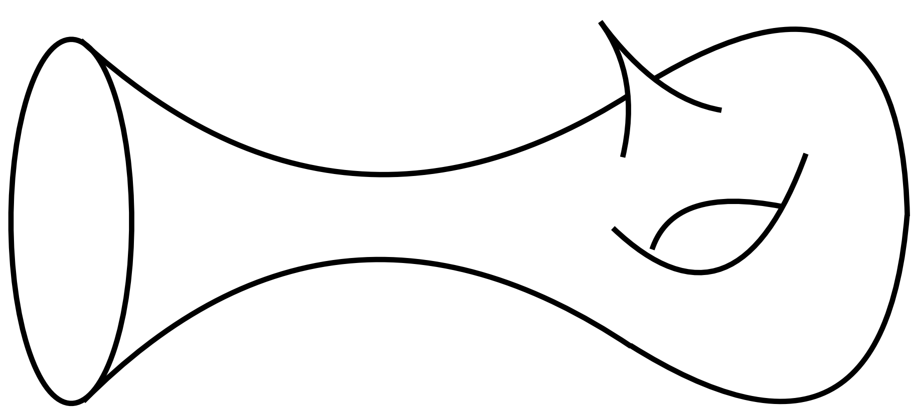
    1. Generalize the dual matrix integral to general dilaton gravity theories

    2. Application to 3D gravity [Maxfield, GJT 20] [Witten 20]

- Repeat the same procedure but allow the presence of dynamical defects. Sum over any number of them and any position.

    - Defect fugacity: λ

    - Deficit angle: $\theta = 2\pi(1-\alpha)$



2D dilaton-gravity

- A defect is equivalent to inserting $\lambda \int \sqrt{g} e^{-2\pi(1-\alpha)\phi}$ in the JT path integral.

[Mertens, GJT 19]

- Then JT gravity with a gas of defects is equivalent to the following modification of the action

$$I = -\frac{1}{2} \int d^2x \sqrt{g} (\phi R + U(\phi))$$

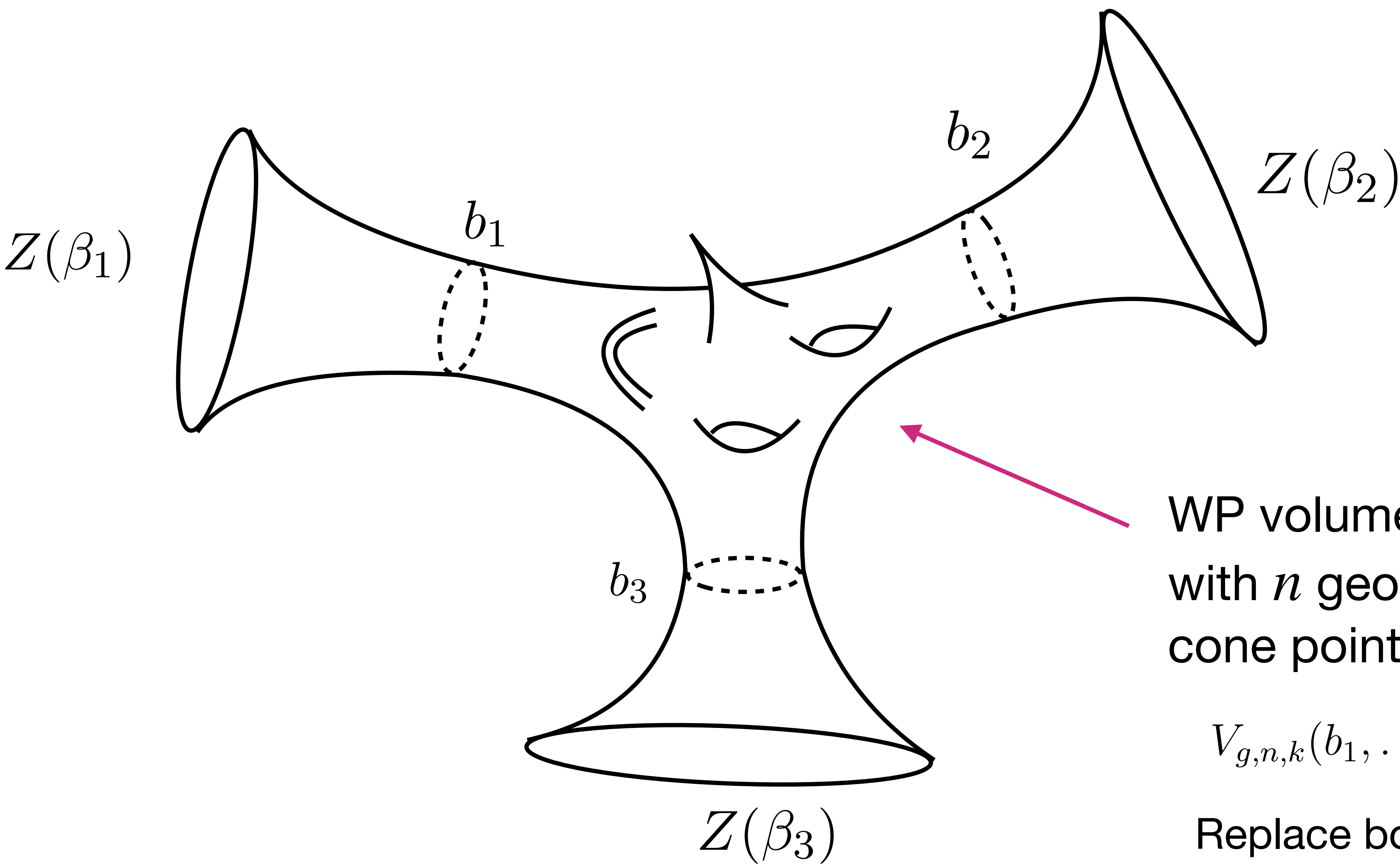
With potential

$$U(\phi) = 2\phi + 2 \sum_i \lambda_i : e^{-2\pi(1-\alpha_i)\phi} :$$

- This covers a large class of two-derivative pure dilaton-gravity.

Cut and glue v2

For defecit angles that satisfy $\alpha < 1/2$ there is always a geodesic homologous to the holographic boundary. Therefore we can still use trumpets to glue. For example



WP volume over hyperbolic surfaces with n geodesic boundaries and k cone points

$$V_{g,n,k}(b_1, \ldots, b_n; \alpha_1, \ldots, \alpha_k) = \mathrm{Vol}\,(\mathcal{M}_{n,g,k})$$

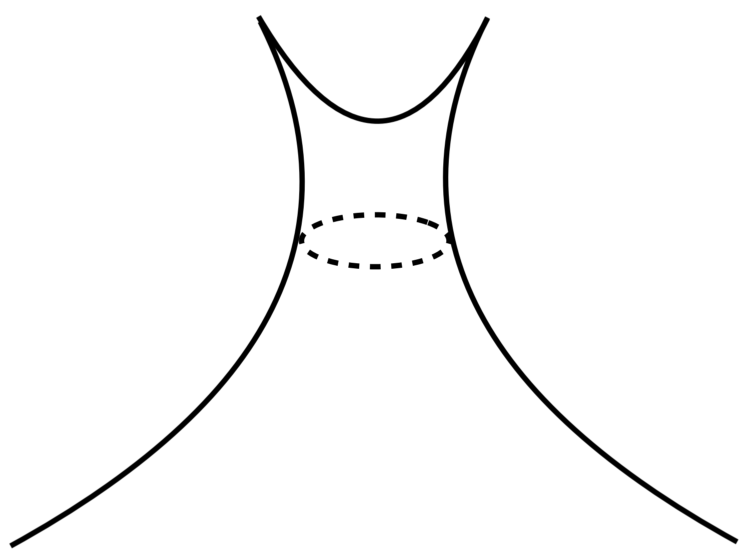
Replace borders by defects $b \to 2\pi i \alpha$

[Tan Wong Zhang] [Do Norbury]

Cut and glue v2

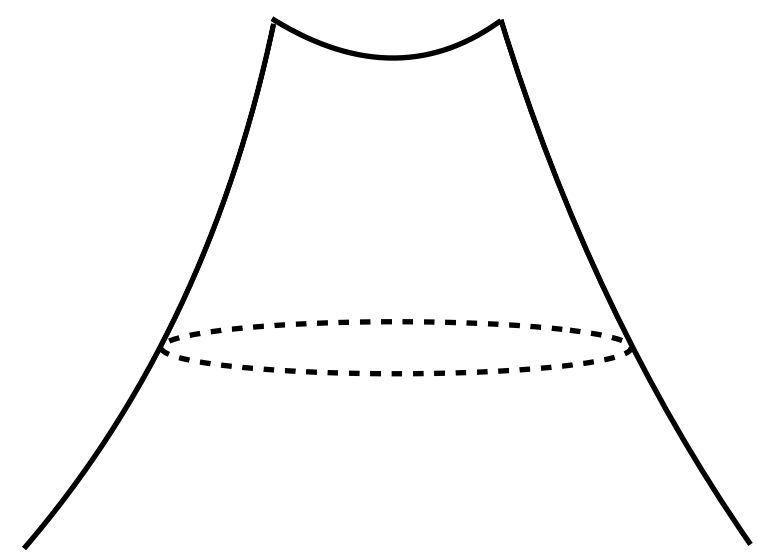
The fact that we restrict to $\alpha < 1/2$ is important. Consider for example the following two situations

$\alpha < 1/2$



Geodesic

$1/2 < \alpha < 1$



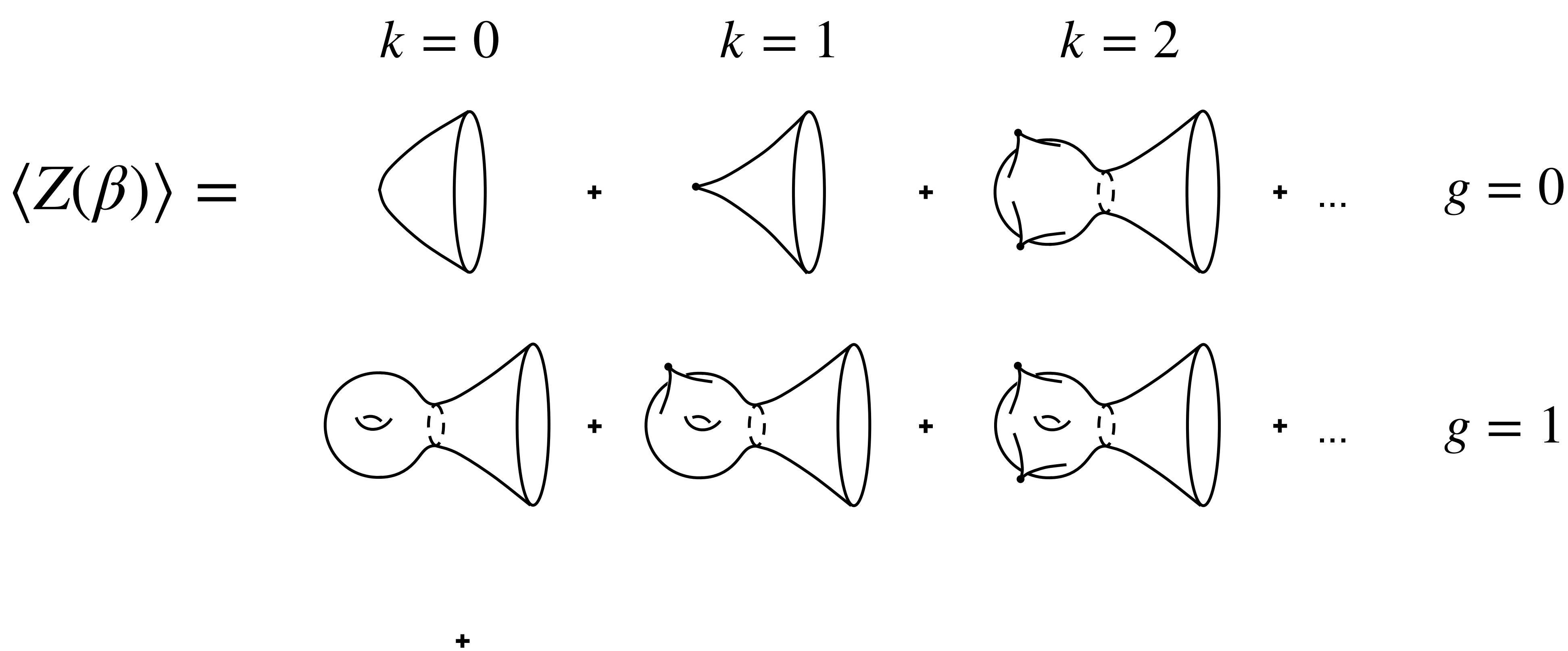
No geodesic!

SSS recipe cannot be applied. We will use a different method later

Now the calculation becomes the same as in JT gravity but with a double expansion. We can also generalize to several flavors of defects:

$$\left\langle Z(\beta_1)\cdots Z(\beta_n)\right\rangle_C = \sum_{g,k_1,k_2,\ldots=0}^{\infty} e^{-(2g+n-2)S_0}\left(\prod_i \frac{\lambda_i^{k_i}}{k_i!}\right) Z_{g,n,k}(\beta_1,\ldots,\beta_n;\alpha_1,\ldots,\alpha_k)$$

For example, in the case of the single boundary:



JT gravity with defects

- Main questions: 1) Can we perform the sum over defects explicitly to get new d.o.s? And 2) Is the theory dual to a matrix integral?

The answer to both questions is **yes**!

- Before, it is instructive to consider the following question. Can we define a theory where we include a finite number of defects? For example, only one.

$$\rho(E) \sim e^{S_0} \left[\sqrt{E} + \frac{\lambda}{\sqrt{E}} - \frac{\lambda^2}{2E^{3/2}} + \dots \right]$$

1 defect   2 defect

No, we **have** to sum over defects

$$\rho(E) \sim e^{S_0} \sqrt{E - E_0}$$

(Including a single defect is analogous to the Maloney Witten partition function in 3D, and its ill-defined for similar reasons)

Genus zero WP volumes

- To compute the genus zero d.o.s. we need to sum over defects. This is done with the following formula for genus zero WP volumes

[Mertens, GJT 20] [Budd wip]

$$V_{0,n}(b_1,\ldots,b_n) = \frac{1}{2}\left(-\frac{\partial}{\partial x}\right)^{n-3}\left[J_0\left(b_1\sqrt{u_{\mathrm{JT}}(x)}\right)\cdots J_0\left(b_n\sqrt{u_{\mathrm{JT}}(x)}\right)u'_{\mathrm{JT}}(x)\right]\Big|_{x\to 0}$$

Where

$$\frac{\sqrt{u_{\mathrm{JT}}}}{2\pi} I_1\left(2\pi\sqrt{u_{\mathrm{JT}}}\right) = x$$

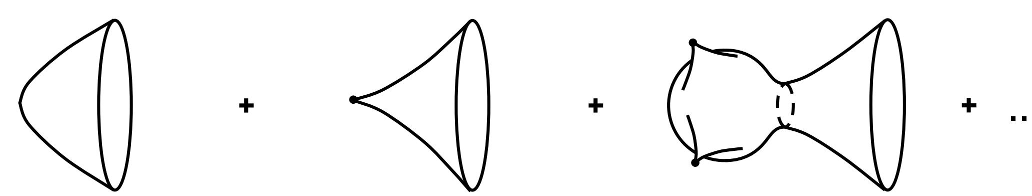
[Zograf 98]

Replace borders by defects $b \to 2\pi i\alpha$

Exact density of states

- Using the previous formula for WP volumes we can compute the disk d.o.s. as

$$\langle \rho(E) \rangle_{g=0} = \frac{e^{S_0}}{4\pi} \int_{E_0}^{E} \frac{du}{\sqrt{E-u}} \left(I_0\left(2\pi\sqrt{u}\right) + \sum_i \lambda_i \frac{2\pi\alpha_i}{\sqrt{u}} I_1\left(2\pi\alpha_i\sqrt{u}\right) \right)$$

$=$ 

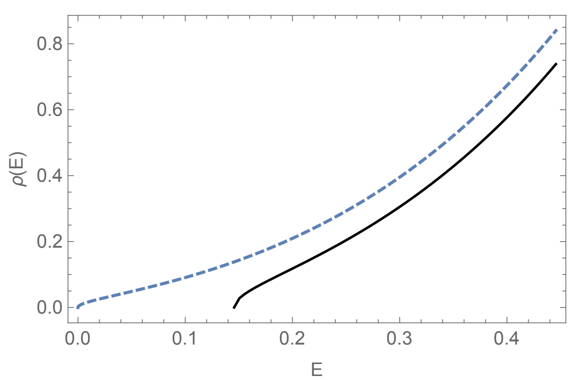
The new edge of the spectrum depends implicitly on the fugacity through

$$\boxed{\sqrt{E_0} I_1\left(2\pi\sqrt{E_0}\right) + 2\pi \sum_i \lambda_i I_0\left(2\pi\alpha_i\sqrt{E_0}\right) = 0}$$

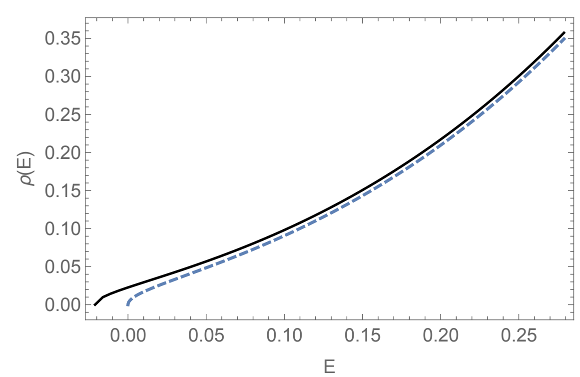
- We can check this matches the previous perturbative calculation.

Exact density of states

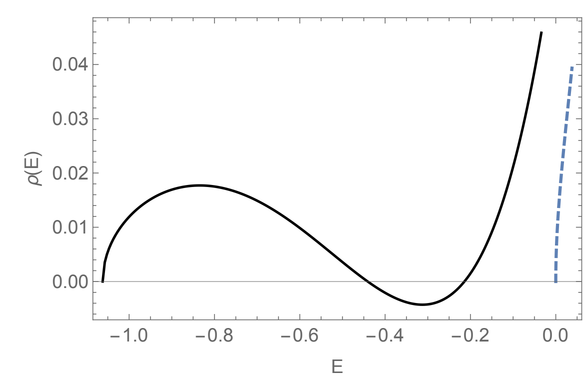
- Some numerical calculation of the density of states:



$\lambda < 0$



$0 < \lambda < \lambda_c$



$\lambda_c < \lambda$

- The theory is perfectly fine for $\lambda < 0$. For $\lambda_c < \lambda$ the density of states can become negative! This critical value is finite.

- The interpretation and fate of the model beyond the critical fugacity is an open question.

Exact answer at $g = 0$ from Gravity

- Previous calculation can be generalized to any number of boundaries

$$Z^{g=0}_{\text{grav}}(\beta_1, \ldots, \beta_n) = \frac{e^{(2-n)S_0}}{2\pi^{n/2}} \frac{\sqrt{\beta_1 \cdots \beta_n}}{\beta_1 + \ldots + \beta_n} \left(\frac{\partial}{\partial x}\right)^{n-2} e^{-u(x)(\beta_1+\ldots+\beta_n)}\Big|_{x=0}$$

With “string equation”: $$\frac{\sqrt{u(x)}}{2\pi} I_1\left(2\pi\sqrt{u(x)}\right) + \lambda\, I_0\left(2\pi\alpha\sqrt{u(x)}\right) = x,$$

- This is the answer for a hermitian matrix integral in the double scaling limit!

[Ambjorn, Jurkiewicz, Makeenko]

[Moore Seiberg Staudacher]

The String Equation

- A matrix integral in the double-scaling limit is specified by coefficients t_k through

[Brezin, Kazakov] [Douglas Shenker] [Gross Migdal]
[Banks Douglas Seiberg Shenker]

$$\sum_k t_k u^k = x,$$

- Related to Disk (genus zero) density of states by:

$$\langle Z(\beta)\rangle_{g=0} = \frac{e^{S_0}}{\sqrt{4\pi\beta}} \int_0^\infty dx e^{-\beta u(x)}$$

- Higher-genus corrections determined by replacing $u^k \to R_k[u; \hbar = e^{-S_0}]$

- JT gravity turns on infinite couplings

$$\frac{\sqrt{u(x)}}{2\pi} I_1\left(2\pi\sqrt{u(x)}\right) + \lambda\, I_0\left(2\pi\alpha\sqrt{u(x)}\right) = x,$$

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Topological Recursion: Deformation Theorem

- Eynard and Orantin showed that under some assumptions, if we have a solution of the topological recursion, the following is also a solution

$$W_{g,n}^{\text{new}}(z's) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \oint_\Gamma dy_1 f(y_1) \cdots \oint_\Gamma dy_k f(y_k) W_{g,n+k}^{\text{old}}(z's; y_1, \ldots, y_k)$$

- For a choice of contour Γ and function $f(y)$ this is precisely the sum over defects

(Important in construction of “Tau” and “Baker-Akhiezer” function, related to integrability)

$$\langle Z(\beta_1)\ldots Z(\beta_n)\rangle_{\text{conn.}} = \sum_{g=0}^{\infty} e^{-(2g-2+n)S_0} Z_{g,n}(\beta_1,\ldots,\beta_n)$$

- **JT gravity + defects:** Computed in terms of WP volumes and includes a sum over defects

- **Matrix Integral:** Computed in terms of the topological recursion of matrix models applied to the new density of states $\rho_{\text{disk}} = \rho_{JT+def}(E)$

Eynard and Orantin "deformation theorem" guarantees both procedures agree!

JT gravity with a gas of defects (or pure dilaton-gravity) is holographically dual to a matrix integral, interpreted as an average over Hamiltonians

2D gravity as a Minimal String

The minimal string theory

- The world-sheet description is

$$\text{Minimal String} = (p, q) \text{ Minimal Model} \quad + \quad \text{Liouville} \quad + \quad \text{Ghosts}$$

$$c_{p,q} + c_{\text{Liouville}} + c_{\text{ghosts}} = 0$$

- We focus on the one-matrix series $(2,p)$ with $p = 3,5,\ldots$ Liouville coupling is $b = \sqrt{2/p}$ [Kazakov] [Staudacher]

- In the limit $p \to \infty$ the theory becomes JT gravity. New twist: interpret the matrix holographically as a dual random Hamiltonian.

[Saad Shenker Stanford]

The disk partition function

- From a worldsheet CFT perspective we need to compute the marked partition function for fixed boundary cosmological constant

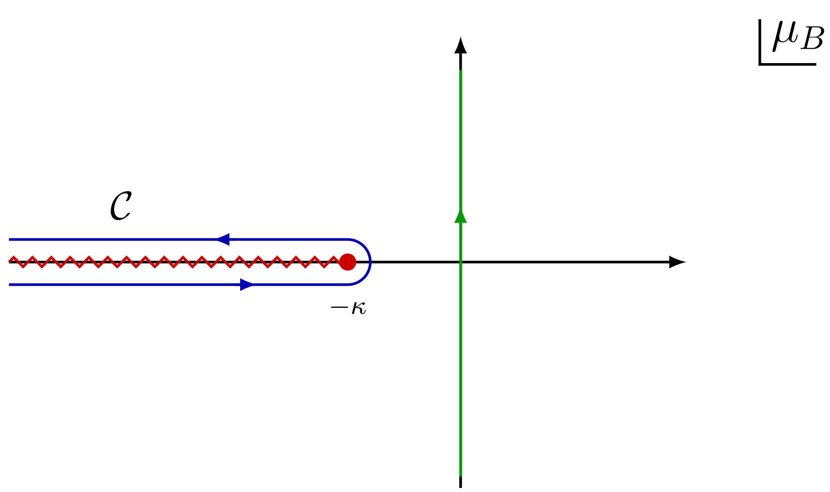
[FZZ, T]
[Seiberg Shih]

$$Z(\mu_B)^{\mathrm{M}} \sim \mu^{\frac{1}{2b^2}} \cosh \frac{2\pi s}{b}, \qquad \mu_B(s) = \kappa \cosh 2\pi b s,$$

- Laplace transform to go to fixed length

$$Z(\ell) \equiv -i \int_{-i\infty}^{i\infty} d\mu_B e^{\mu_B \ell} Z(\mu_B)^{\mathrm{M}}$$

- Rotate contour



- Final answer: Allows us to extract leading density of states of MM

$$Z(\ell) \sim \mu^{\frac{1}{2b^2}+\frac{1}{2}} \int_0^\infty ds\ e^{-\ell\kappa\cosh(2\pi b s)} \rho(s), \qquad \rho(s) \equiv \sinh 2\pi b s \sinh \frac{2\pi s}{b}.$$

Minimal String and JT gravity

	JT gravity	Minimal String
DOS	$\frac{1}{4\pi^2}\sinh\left(2\pi\sqrt{E}\right)$	$\frac{1}{4\pi^2}\sinh\left(\frac{p}{2}\text{arccosh}\left(1+\frac{8\pi^2}{p^2}E\right)\right)$
String equation	$\frac{\sqrt{u}}{2\pi}I_1(2\pi\sqrt{u}) = x$	$\frac{u}{2}F\left(\frac{1-p}{2},\frac{1+p}{2},2,-\frac{4\pi^2}{p^2}u\right) = x$

$$\xleftarrow[p\to\infty]{}$$

[Okuyama Sakai] [Betzios Papadoulaki] [Johnson] [Mertens GJT]…

$$\langle Z(\beta)\rangle_{g=0} = \frac{e^{S_0}}{\sqrt{4\pi\beta}}\int_0^\infty dx e^{-\beta u(x)}$$

A sketch of a proof

- Describe the minimal string with Liouville field ϕ and the minimal model with a Coulomb gas (time-like Liouville) field χ

- Define a new metric $g = e^{2\rho}\hat{g}$ and dilaton Φ by
  [Seiberg Stanford]
  See also: [Kyono Okumura Yoshida 17]

$$\phi = b^{-1}\rho + b\Phi$$
$$\chi = b^{-1}\rho - b\Phi$$

- The action becomes 2D dilaton-gravity:

$$I = -\frac{1}{2}\int d^2x\sqrt{g}\left(\Phi R + 2p\sinh p^{-1}\Phi\right)$$

- The limit $p \to \infty$ automatically gives JT gravity

Deformation of the Minimal String

	JT gravity + defects	Minimal String
String equation	$\frac{\sqrt{u}}{2\pi} I_1(2\pi\sqrt{u}) + \sum_i \lambda_i I_0(2\pi\alpha_i\sqrt{u}) = x$	**?**

An example: One defect

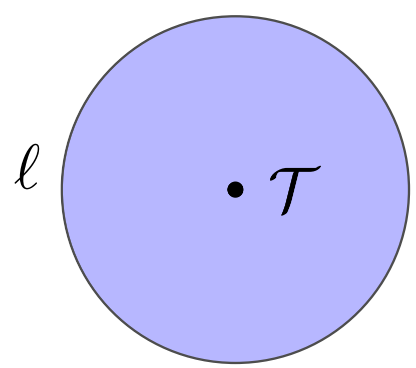
[Mertens GJT]

- We can compute the partition function to leading order in the deformation

- From a worldsheet CFT perspective we need to compute $\mathcal{T} = \int_\Sigma d^2z\, \mathcal{O}_{\mathrm{M}}(z,\bar{z}) e^{2\alpha\phi(z,\bar{z})}$

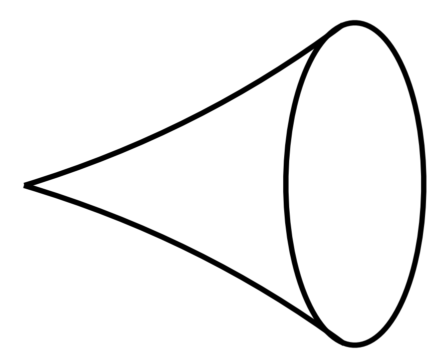
[FZZ, T]

$$\langle \mathcal{T}_{\alpha_M} \rangle_\ell =$$



$$\langle \mathcal{T} \rangle \sim \int_0^\infty ds\; e^{-\ell \mu_B(s)} \cos 4\pi P s$$

- The result for this bulk one-point function matches with the one defect amplitude in the disk (in the JT limit)



From tachyons to defects

[Usatyuk, Weng, GJT]

- Consider the following perturbation of the minimal string

$$I = I_{(2,p)} + \sum_n \tau_n \int e^{2\alpha_n \phi} \mathcal{O}_{1,n}$$

- In the dilaton-gravity description this becomes

$$I = -\frac{1}{2} \int \sqrt{g} \left[\Phi R + 2U(\Phi) \right], \qquad \Rightarrow \qquad U(\Phi) = 2\mu \sinh\left(2\pi b^2 \Phi\right) + \sum_{n=1}^{m-1} \tau_n \, e^{-2\pi b^2 n \Phi},$$

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Defect with angle

$$n = (1-\alpha)/b^2$$

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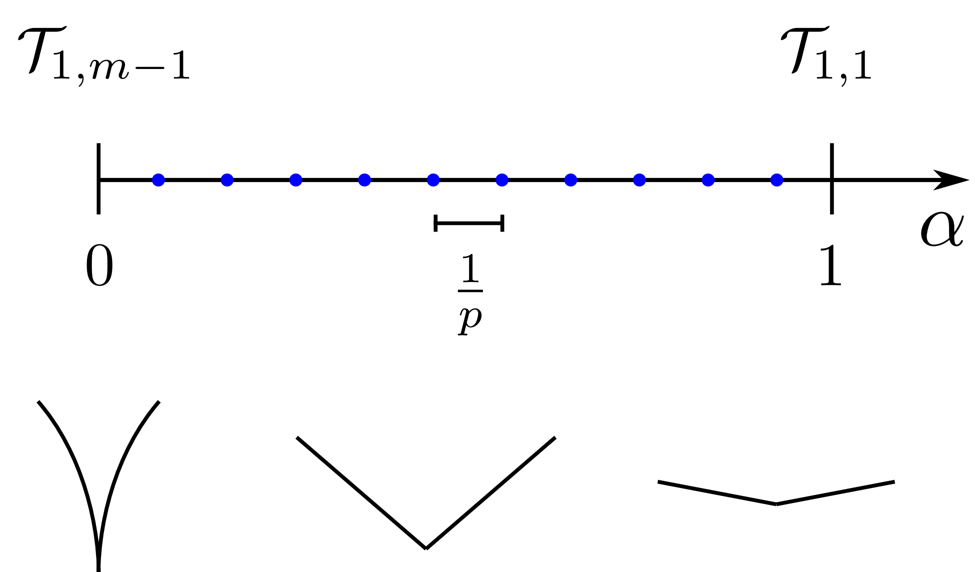
- Spectrum of defects:

$$n = (1-\alpha)/b^2$$

$$p \to \infty \ (b \to 0)$$



Deformed Minimal String: Exact Solution

- Deform the minimal string action by: $$I = I_{(2,p)} + \sum_n \tau_n \int e^{2\alpha_n \phi} \mathcal{O}_{1,n}$$

- Can we find the string equation of this theory? This amounts to solving the theory exactly (assuming it is dual to matrix integral)

$$t_k(\tau_n, \mu)?$$

Deformed Minimal String: Exact Solution

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[Moore Seiberg Staudacher 91] [Belavin Zamolodchikov 08]

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- Bootstrap approach: Correlators in sphere should satisfy fusion rules of minimal model CFT [Moore Seiberg Staudacher 91] [Belavin Zamolodchikov 08]

- Exact answer for "sharp" deformations:

$$\mathcal{P}_0(u) + \sum_n \tau_n P_{\frac{p-1}{2}-n}\left(1 + \frac{8\pi^2}{p^2} u\right) = x$$

Undeformed minimal string

Deformation of the Minimal String

[Usatyuk, Weng, GJT]

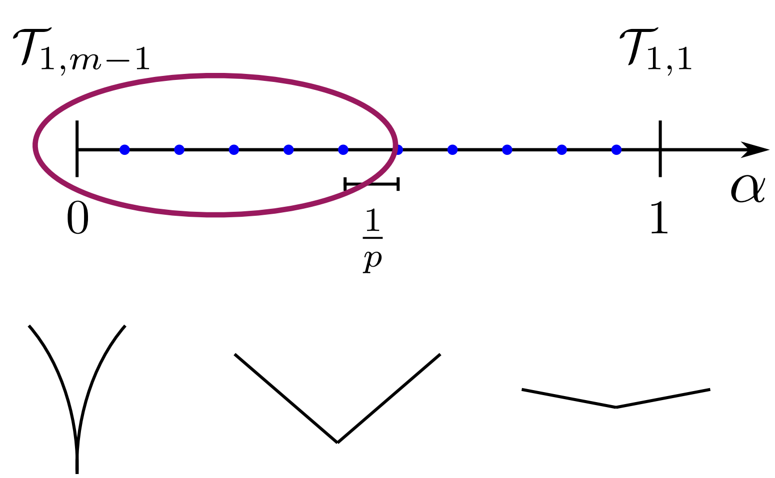
- We can check that in the large p limit this reduces to JT coupled to sharp defects:

	JT gravity + defects	Minimal String
String equation	$\frac{\sqrt{u}}{2\pi} I_1(2\pi\sqrt{u}) + \sum_i \lambda_i I_0(2\pi\alpha_i\sqrt{u}) = x$	$\mathcal{P}_0(u) + \sum_n \tau_n P_{\frac{p-1}{2}-n}\left(1 + \frac{8\pi^2}{p^2}u\right) = x$

$\xleftarrow{\;p\to\infty\;}$

- Valid in the range:

$\alpha < 1/2$



- Belavin and Zamolodchikov also conjectured a solution for deformations with small n, looks a little uglier:

Undeformed ⟶

$$\frac{p}{16\pi^2}\left(P_m\left(\frac{u_{\rm MS}}{\kappa}\right)-P_{m-2}\left(\frac{u_{\rm MS}}{\kappa}\right)\right)$$
$$+\sum_{L=1}^{\infty}\sum_{n_1,\ldots,n_L=1}^{m-1}\frac{1}{L!}\Pi_{i=1}^{L}\lambda_{n_i}\left(\frac{16\pi^2}{p^2}\right)^{L-1}P^{(L-1)}_{m-1-\sum_{i=1}^{L}n_i}\left(\frac{u_{\rm MS}}{\kappa}\right)=x,$$

Notation: $p=2m-1$, $\kappa\propto\sqrt{\mu}$, $u_{MS}=\kappa\left(1+\frac{8\pi^2}{p^2}u\right)$ and $\lambda\propto\tau$

- When n is large, only $L=1$ remains, and we recover sharp defects, since we have a constrain

$$\sum_{i=1}^{L} n_i < m-1$$

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➡ We can take the large p limit of this formula and obtain a solution to 2D dilaton-gravity with generic defects!

➡ This works when the cut-and-glue prescription of SSS cannot be implemented

Example: One generic defect

[Usatyuk, Weng, GJT]

- String equation from the minimal string

$$\frac{\sqrt{u}}{2\pi} I_1(2\pi\sqrt{u}) + \sum_{L=1}^{\lfloor \frac{1}{1-\alpha} \rfloor} \frac{\lambda^L}{L!} \left(\frac{2\pi(1 - L(1-\alpha))}{\sqrt{u}} \right)^{L-1} I_{L-1}\left(2\pi(1 - L(1-\alpha))\sqrt{u}\right) = x$$

- Outside the SSS framework (no geodesics) but still dual to matrix integral

- Includes merging L defects $\alpha_L = 1 - L(1-\alpha)$.

- Non-trivial check: gives back JT when $\alpha \to 1$

Example: One generic defect

[Usatyuk, Weng, GJT]

- String equation

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- A check: when we take $\alpha = 1$ we should recover JT

$$\frac{\sqrt{u}}{2\pi} I_1(2\pi\sqrt{u}) + \sum_{L=1}^{\infty} \frac{\lambda^L}{L!} \left(\frac{\sqrt{u}}{2\pi} \right)^{1-L} I_{L-1}(2\pi\sqrt{u}) = \frac{\sqrt{u+2\lambda}}{2\pi} I_1(2\pi\sqrt{u+2\lambda})$$

Gives back JT with a shift of energy

Solution of dilaton-gravity on disk

[Usatyuk, Weng, GJT]

[Budd, wip]

- General solution for density of states

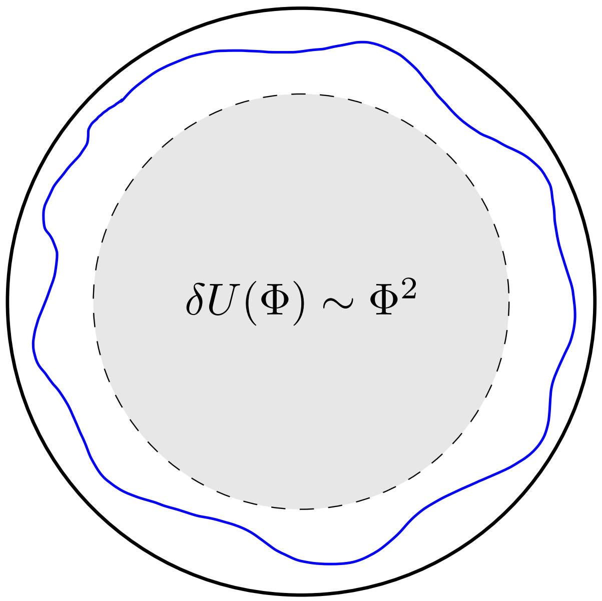
$$\rho(E) = \frac{e^{S_0}}{2\pi} \int_{\mathcal{C}} \frac{dy}{2\pi i} \, e^{2\pi y} \, \tanh^{-1}\left(\sqrt{\frac{E-E_0}{y^2 - 2W(y) - E_0}}\right)$$

With defect potential:

$$W(y) \equiv \sum_i \lambda_i e^{-2\pi(1-\alpha_i)y}$$

- Using defects with $\alpha \sim 1$ we can model more general dilaton potentials

The exact formula matches with semiclassical limit



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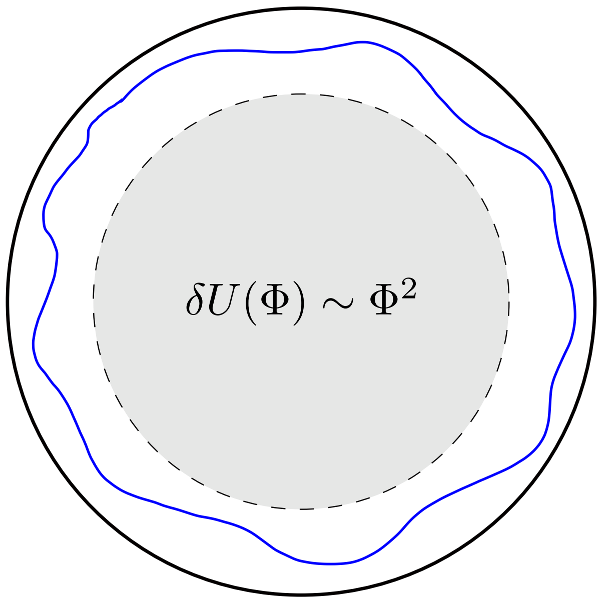

The exact formula matches with semiclassical limit

(There is a puzzle I don't have time to mention)

Conclusions

- We solved pure 2D dilaton-gravity

- We argued that all these theories are dual to a matrix integral, in a holographic way

- Connection to minimal string

- Open Questions:

  - Dual of 2D dilaton-gravity with matter?
  - Finite cut-off AdS/ relation with $T\bar{T}$? [Iliesiu Kruthoff Verlinde GJT] [Stanford Yang]
  - MM of JT from "triangulation" perspective? [Kazakov Staudacher Wynter]
  - Relation between minimal string and SYK? [Berkooz Isachenkov Narovlansky Torrents]
  - We worked in AdS, what about flat space and dS?