Exact g-functions

based on 2004.05071 with Shota Komatsu

João Caetano
(SCGP & YITP)

London Integrability Journal Club
June 25, 2020

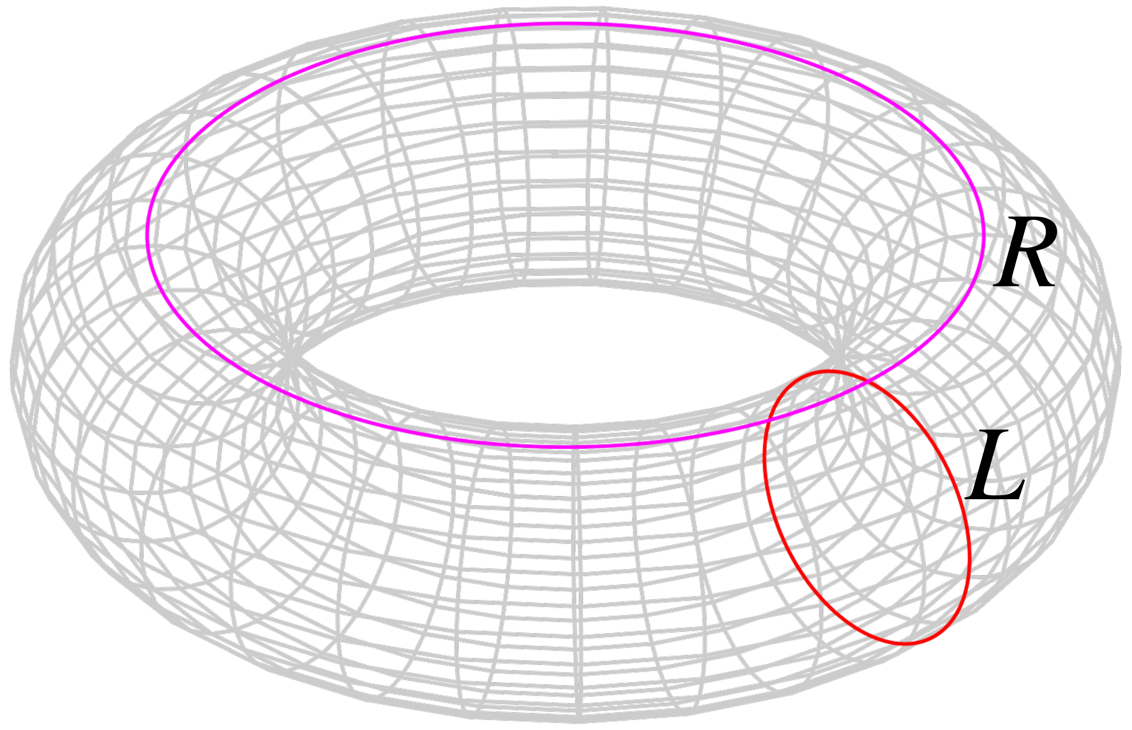
SIMONS CENTER
FOR GEOMETRY AND PHYSICS

Functional Equations in Integrable Field Theories

Asymptotic Bethe Ansatz (periodic)

$$e^{ip_iL} = \prod_{j\neq i}^{N} S(p_i, p_j) \qquad E = \sum_j \epsilon(p_j) + \mathcal{O}(e^{-L})$$

Finite volume: theory on a torus ($T = 1/R$)



Saddle point $R \to \infty$: $\quad Z \sim e^{-RE_0(L)}$

Swap space $\leftrightarrow$ time: $\quad E_0 = Lf$

Free energy density at finite temperature:
Thermodynamic Bethe Ansatz

Other quantities: form factors, correlation functions etc.

no such prescription: $\mathcal{O}(1)$ pieces of partition function, or partition function with insertions...

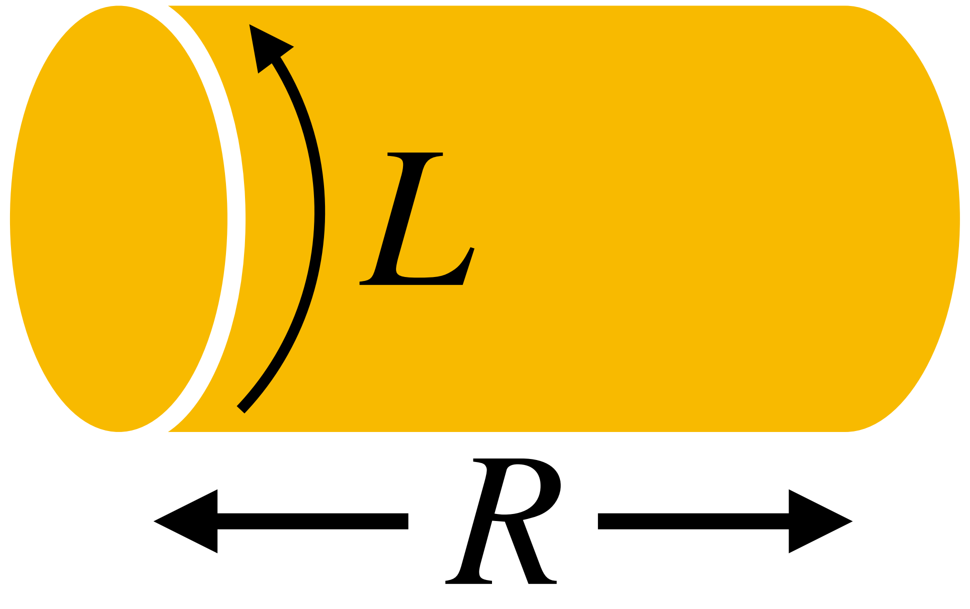
g-function

- Next to simplest quantity: simplest generalization of the analysis of the spectrum

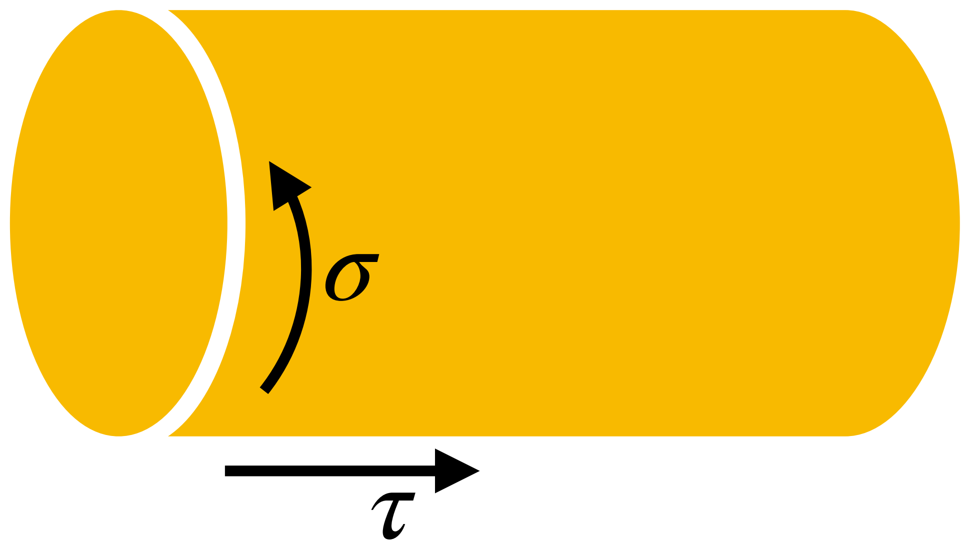
- $\mathcal{O}(1)$ quantity exactly computable in any integrable field theory at finite volume.

- No TBA directly for the g-function.

g-function

$Z = \langle B |$  $| B \rangle$

Closed string channel:



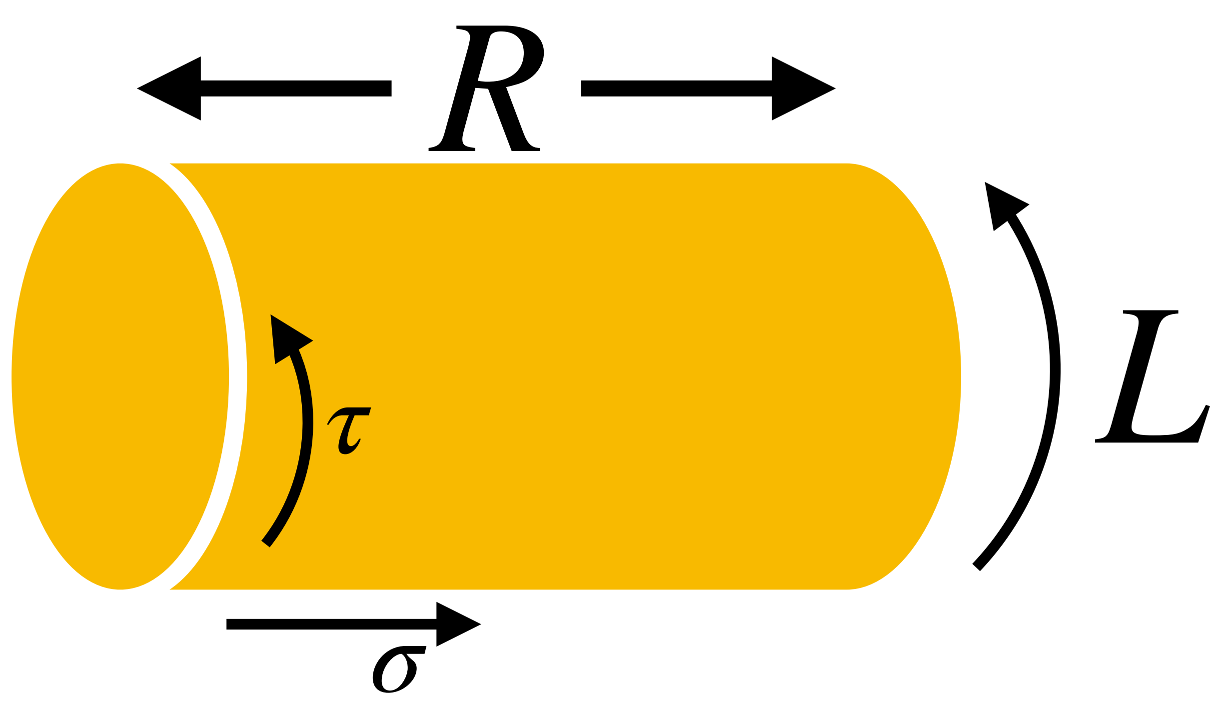
$$Z = \sum_{\psi_c} e^{-RE_{\psi_c}(L)} \frac{\langle B | \psi_c \rangle \langle \psi_c | B \rangle}{\langle \psi_c | \psi_c \rangle}$$

As $R \to \infty$ $\quad Z \sim e^{-RE_\Omega} | g |^2$

g-function: $g \equiv \dfrac{\langle B | \Omega \rangle}{\sqrt{\langle \Omega | \Omega \rangle}}$

Also known as ground state degeneracy or boundary entropy

Open string channel:



$$Z = \sum_{\psi_o} e^{-LE_{\psi_o}(R)}$$

$$e^{-RE_\Omega}|g|^2 = \lim_{R\to\infty} \sum_{\psi_o} e^{-LE_{\psi_o}(R)}$$

Thermal partition function in infinite volume at finite temperature $1/L$

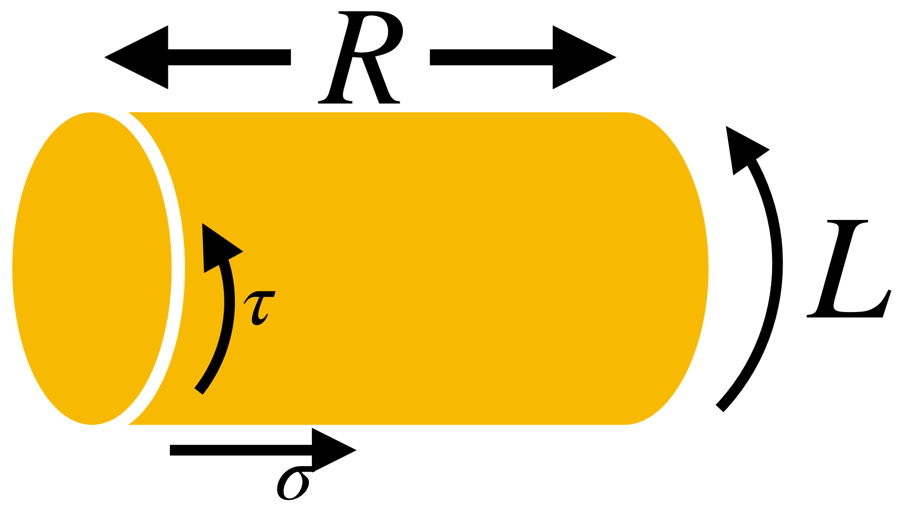
Outline

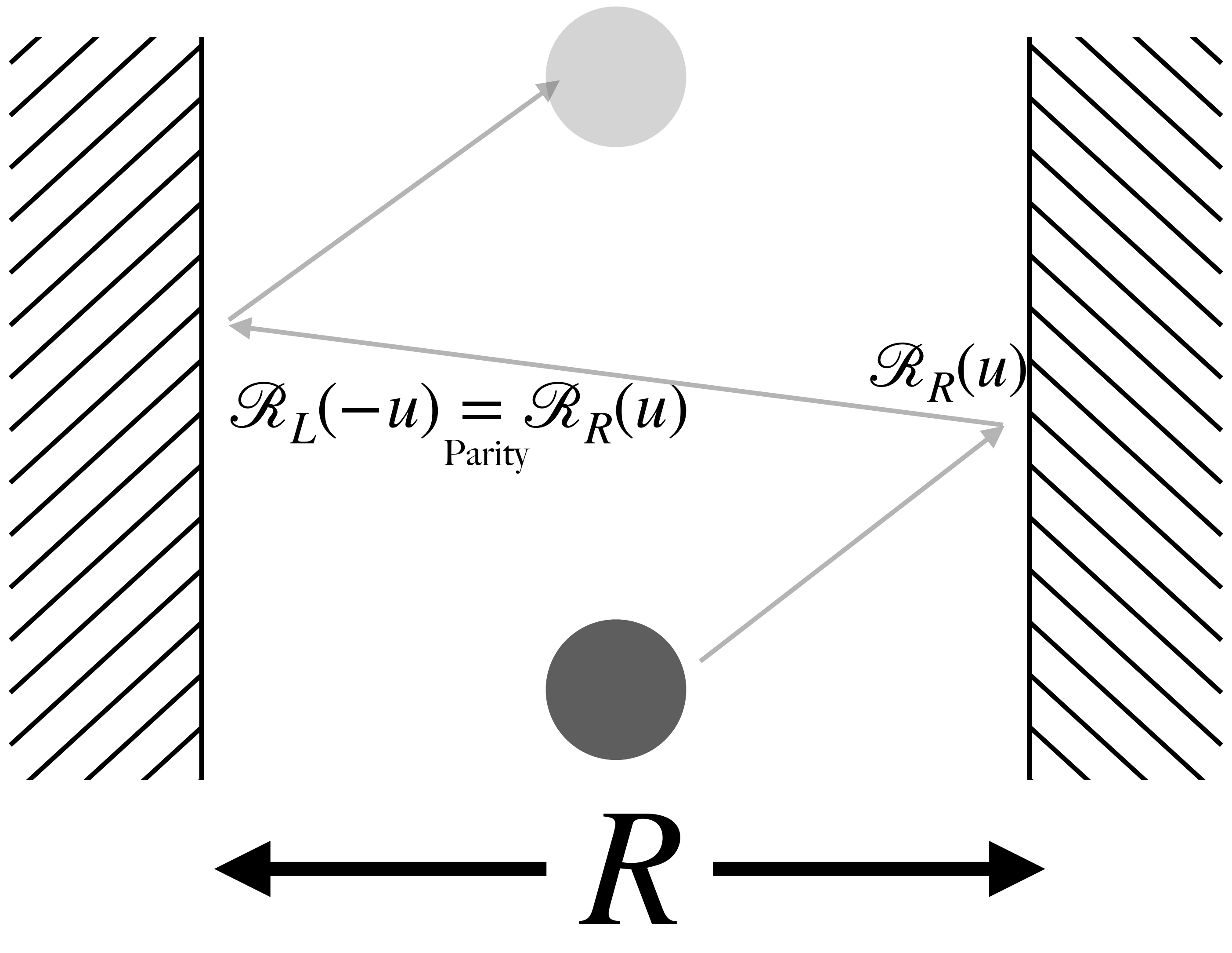
- g-function as a Fredholm determinant from TBA
- Tracy-Widom TBA for Sinh-Gordon
- UV limit and Liouville FZZT states
- Separation of Variables
- Outlook

Exact g-function = Fredholm Det

- [A. LeClair, G. Mussardo, H. Saleur and S. Skorik '95]: First attempt.
- [F. Woynarovich '04] pointed out incompleteness of the previous and proposed modification.
- [P. Dorey, D. Fioravanti, C. Rim and R. Tateo '04] proposed yet another modification to the previous.
- [B. Pozsgay'10] verified and re-derived previous result from a different approach.
- [I. Kostov, D. Serban and D.-L. Vu ' 18] rigorously re-derived previous result.
- [I. Kostov ' 19] re-re-derived previous result as an effective QFT whose path integral can be localized.
- [Jiang, Komatsu, Vescovi' 20] offers yet another derivation.

Open string channel





Massive relativistic theory with single type particle :

$$E = m\cosh u \quad p = m\sinh u$$

$$e^{2im\sinh(u_i)R}\left[\prod_{j\neq i}^{N} S(u_i - u_j)S(u_i + u_j)\right]\mathscr{R}_a(u_i)\mathscr{R}_b(u_i) = 1$$

$$u_i > 0$$

$$e^{2im\sinh(u_i)R}\left[\prod_{j\neq i}^{2N} S(u_i - u_j)\right]\mathscr{R}_{ab}(u_i) = 1 \qquad u_{2N-j} = -u_j$$

$$\mathscr{R}_{ab}(u) \equiv \mathscr{R}_a(u)\,\mathscr{R}_b(u)\,S(2u)$$

Thermodynamic limit

$$e^{2im\sinh(u_i)R}\left[\prod_{j\neq i}^{2N} S(u_i-u_j)\right]\mathscr{R}_{ab}(u_i)=1$$

Rapidity density: $$2\pi(\rho^{\mathrm{o}}(u)+\rho^{\mathrm{h}}(u))=m\cosh u-2\pi\int_{-\infty}^{\infty}dv\,\mathscr{K}_s(u-v)\rho^{\mathrm{o}}(v)+\frac{\Theta_{ab}}{2R}$$

$$\Theta_{ab}(u)=\frac{1}{i}\frac{d}{du}\log\left(\mathscr{R}_a(u)\mathscr{R}_b(u)\right)-\frac{1}{i}\frac{d}{du}\log(S(2u))-2\pi\delta(u)$$

Z at the saddle point: $$\log Z\simeq\frac{1}{4\pi}\int_{-\infty}^{\infty}du\left(2mR\cosh u+\Theta_{ab}(u)\right)\log(1+e^{-\epsilon(u)})$$

R-independent

saddle point equation (closed system TBA): $$\epsilon(u)=mL\cosh(u)+\int_{-\infty}^{\infty}dv\,\mathscr{K}_s(u-v)\log(1+e^{-\epsilon(v)})$$

Open-closed channel match: g-function

$$e^{-RE_\Omega} |g|^2 = \lim_{R\to\infty} \sum_{\psi_o} e^{-LE_{\psi_o}(R)}$$

$$\log(g) = \frac{1}{4\pi} \int_{-\infty}^{\infty} du\, \Theta_{ab}(u) \log(1 + Y(u))$$

[LeClair, Mussardo, Saleur, Skorik '95]

Incomplete!

- Assumption: $\sum_{\psi_o} \to \int R\Delta u\, d\rho_o(u)$

- Neglected fluctuations around the saddle point

Corrections

$$\sum_{\psi_o} = \mathcal{N} \int R\Delta u \, d\rho_o(u)$$

Jacobian for the transformation of
momentum quantum numbers to rapidities

$$Z = \mathrm{Det}(1 - \hat{G})^{1/2} \, \mathrm{Det}(1 - \hat{G}_+)^{-1} e^{-RF_{\text{saddle}}}$$

quadratic fluctuations
around the saddle point [Woynarovich '04]

Det = Fredholm Determinants

Final expression

$$\log g = \int_0^\infty \frac{du}{2\pi}\Theta(u)\log(1+e^{-\epsilon(u)}) + \frac{1}{2}\log\frac{\mathrm{Det}(1-\hat{G})}{\left(\mathrm{Det}(1-\hat{G}_+)\right)^2}.$$

$$\hat{G}_+\cdot f(u) := \int_0^\infty \frac{dv}{2\pi}\frac{\mathscr{K}_+(u,v)}{1+e^{\epsilon(v)}}f(v), \qquad \hat{G}\cdot f(u) := \int_{-\infty}^\infty \frac{dv}{4\pi}\frac{\mathscr{K}_s(u,v)}{1+e^{\epsilon(v)}}f(v)$$

$$\mathscr{K}_+(u,v) = \frac{1}{i}(\partial_u \log S(u,v) + \partial_u \log S(u,-v))$$

$$\mathscr{K}_s(u,v) = \frac{1}{i}(\partial_u \log S(u,v))$$

Fredholm determinants

Representation as a multiple integral:

$$\log \frac{\mathrm{Det}(1-\hat{G})}{\left(\mathrm{Det}(1-\hat{G}_+)\right)^2} = \sum_{n=1}^{\infty} \frac{1}{n} \int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{du_i}{2\pi} \frac{1}{1+e^{\epsilon(u_i)}} \mathscr{K}_s(u_1+u_2) \prod_{j=2}^{n} \mathscr{K}_s(u_j - u_{j+1}),$$

Not very efficient, especially if one is aiming at generalizations:

- Nesting [P. Dorey, A. Lishman, C. Rim, R. Tateo '05, I. Kostov, D. Serban and D.-L. Vu '19, Jiang, Komatsu, Vescovi '20];
- Excited states [I. Kostov, D. Serban and D.-L. Vu '19, Jiang, Komatsu, Vescovi '20]
- 3-point functions with two Giant Gravitons and a non-BPS single trace in $\mathcal{N}=4$ SYM given as a g-function: involves nesting, excited states etc. [Jiang, Komatsu, Vescovi '20]

Goal: derive a TBA to compute these Fredholm determinants

Towards functional equations

Relation between Fredholm determinants and TBA:

- $\mathcal{N} = 2$ supersymmetric index in two dimensions [Cecotti, Fendley, Intriligator, Vafa '92]
- The partition function of 2d polymers [Zamolodchikov]
- Relations proven in Tracy-Widom '94
- S^3 partition functions supersymmetric gauge theories [Calvo, Grassi, Hatsuda, Marino, Moriyama, Okuyama...]

Two layered system:

Solve standard TBA
with source $m\cosh(u)$
to get $Y(u) \equiv e^{-\epsilon(u)}$

↓

Tracy-Widom TBA
with source $Y(u)$ to get g

Simplest example: sinh-Gordon

One single type of particle of mass m $$\mathscr{L} = \frac{1}{4\pi}(\partial\phi)^2 + 2\mu\cosh(2b\phi)$$

$$S(u,v) = \frac{\sinh(u-v) - i\sin(\pi p)}{\sinh(u-v) + i\sin(\pi p)}$$ $$p = b^2(1+b^2)^{-1}$$

Consider self-dual point $b = 1$, for which $\mathscr{K}_s(u,v) \sim \dfrac{1}{\cosh(u-v)}$

Boundary Sinh-Gordon (open string): $$\mathscr{L} = \left(\frac{1}{4\pi}(\partial\phi)^2 + 2\mu\cosh(2b\phi)\right) + 2\mu_B\left(\cosh(b\,\phi - b\phi_0)\,|_{x=0} + \cosh(b\,\phi - b\phi_0)\,|_{x=R}\right)$$

Also integrable, and reflection matrices are known

Class of kernels

Derivation in principle valid for any kernel of the type: $$K(u,v) = \frac{E(u)\,E(v)}{M(u)+M(v)}$$

Use $\mathrm{Det}(\ldots) = e^{\mathrm{tr}\log(\ldots)}$

$$\begin{aligned}\mathrm{Det}(1 - z\,\hat{G}) &= \exp\left(-\sum_n \frac{z^n}{n}\int\prod_i du_i\,\frac{\mathcal{K}_s(u_i,u_{i+1})}{1+e^{\epsilon(u_i)}}\right)\\ &= \exp\left(-\sum_n \frac{z^n}{n}\int\prod_i du_i K_s(u_i,u_{i+1})\right)\\ &\equiv \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{tr}\,K_s^{*n}\right)\end{aligned}$$

$$K_s(u,v) \equiv \frac{\mathcal{K}_s(u,v)}{\sqrt{1+e^{\epsilon(u)}}\sqrt{1+e^{\epsilon(v)}}}$$

$$E(u) \equiv \frac{\sqrt{2}\,e^u}{\sqrt{1+e^{\epsilon(u)}}} \qquad M(u) \equiv e^{2u}$$

In the case of K_+: $$E(u) \equiv \frac{\sqrt{2}\,\cosh u}{\sqrt{1+e^{\epsilon(u)}}} \qquad M(u) \equiv \cosh(2u)$$

Derivation of Tracy-Widom TBA

Start by deriving a recursion relation for K^{*n}

$$K(u,v) = \frac{E(u)\,E(v)}{M(u)+M(v)}$$

Interpret $E(u)$ as a sort of "wave-function" : $\langle u \,|\, E \rangle = E(u)$

Define $\hat{M}$ as an operator: $\hat{M}\,|\,u\rangle = M(u)\,|\,u\rangle$

Then the kernel becomes: $\hat{M}\hat{K} + \hat{K}\hat{M} = |\,E\rangle\langle E\,|$

For higher powers of $\hat{K}$

$$\begin{aligned}\hat{M}\hat{K}^2 - \hat{K}^2\hat{M} &= (\hat{M}\hat{K} + \hat{K}\hat{M})\hat{K} - \hat{K}(\hat{M}\hat{K} + \hat{K}\hat{M}) \\ &= |\,E\rangle\langle E\,|\,\hat{K} - \hat{K}\,|\,E\rangle\langle E\,|\end{aligned}$$

Recursively:

$$\hat{M}\hat{K}^n - (-1)^n\hat{K}^n\hat{M} = \sum_{l=0}^{n-1}(-1)^l\hat{K}^l\,|\,E\rangle\langle E\,|\,\hat{K}^{n-1-l}$$

$$\hat{M}\hat{K}^n - (-1)^n \hat{K}^n \hat{M} = \sum_{l=0}^{n-1} (-1)^l \hat{K}^l |E\rangle\langle E| \hat{K}^{n-1-l}$$

Sandwich both sides with $\langle u|$ and $|v\rangle$

$$K^{*n}(u,v) = \frac{E(u)E(v)}{M(u) + (-1)^{n-1}M(v)} \sum_{l=0}^{n-1} (-1)^l \underbrace{\langle u|\hat{K}^l|E\rangle}_{\equiv\, \phi_l(u)} \underbrace{\langle E|\hat{K}^{n-l-1}|v\rangle}_{\equiv\, \phi_{n-l-1}(v)}$$

With $\phi_j(u) = \frac{1}{E(u)} \int dv\, K(u,v)\, E(v) \phi_{j-1}(v) \qquad \phi_0(u) = 1$

Baxter-like equations

$$\phi_j(u) = \frac{1}{E(u)} \int dv\, K(u,v)\, E(v) \phi_{j-1}(v) \qquad \phi_0(u) = 1$$

$$K(u,v) \sim \frac{1}{\cosh(u \pm v)}$$

Use $$\frac{1}{\cosh\left(u + \frac{i(\pi-\epsilon)}{2}\right)} + \frac{1}{\cosh\left(u - \frac{i(\pi-\epsilon)}{2}\right)} = 2\pi\delta(u)$$

Shift by $\pm i\pi/2$

To get something like $$\tilde{\phi}_j^{++} + \tilde{\phi}_j^{--} = \frac{2\pi}{1+e^{\epsilon}} \tilde{\phi}_{j-1}$$

$$\tilde{\phi}_j = \frac{\sqrt{1+e^{\epsilon(u)}}}{\sqrt{2}} E(u)\, \phi_j$$

Sum on both sides $\sum_j z^j(\ldots)$

$$P^{++} + P^{--} = 2\pi\, v\, z\, Q$$
$$Q^{++} + Q^{--} = 2\pi\, v\, z\, P$$

$$v = \left(1+e^{\epsilon}\right)^{-1/2} \left(1+e^{\epsilon^{++}}\right)^{-1/2}$$

$$P(u) \propto \sum_{j=0}^{\infty} z^{2j+1} \phi_{2j+1}(u)$$

$$Q(u) \propto \sum_{j=0}^{\infty} z^{2j} \phi_{2j}(u)$$

Back to the kernel

Split the kernel into odd & even parts:

$$\hat{R}_{\mathrm{o}} \equiv \hat{K}(I - z^2\hat{K}^2)^{-1}\,, \quad \hat{R}_{\mathrm{e}} \equiv \hat{K}^2(I - z^2\hat{K}^2)^{-1}$$

The kernels can then be expressed in terms of the Baxter functions Q, P

$$R_{\mathrm{o}}(u,v) = \frac{Q(u)Q(v) - P(u)P(v)}{M(u) + M(v)} \quad R_{\mathrm{e}}(u,v) = \frac{Q(u)P(v) - Q(v)P(u)}{M(u) - M(v)}$$

Goal: derive a closed system of equations for R_o and R_e.

Closed system of equations

R_o and R_e only: no closed system of equations

Need one additional function η

$$\eta(u) - i \equiv -i \frac{(Q^+ - P^+)(Q^- + P^-)}{E^+ E^-}$$

$$R_{\rm o}(u) = \lim_{v \to u} \frac{Q(u)Q(v) - P(u)P(v)}{M(u) + M(v)}$$

$$R_{\rm e}(u) = \lim_{v \to u} \frac{Q(u)P(v) - Q(v)P(u)}{M(u) - M(v)}$$

$+$

Baxter-like equations

$$P^{++} + P^{--} = 2\pi\, v\, z\, Q$$

$$Q^{++} + Q^{--} = 2\pi\, v\, z\, P$$

"Y-system" for Tracy-Widom TBA

For $\mathcal{K}_s$:

$$\log(1+\eta^2) = \log(1+e^{\epsilon^+}) + \log(1+e^{\epsilon^-}) + \log R_e^+ + \log R_e^-$$

$$\frac{2i\,\eta'}{\eta^2+1} = 2i\arctan(\eta)' = \frac{R_o^+}{R_e^+} - \frac{R_o^-}{R_e^-}$$

$$\eta^+ + \eta^- = 2\pi R_e(u)$$

Invert to obtain Tracy-Widom TBA:

For $\mathcal{K}_s$:

$$\eta_s = 2\int_{-\infty}^{\infty} dv \frac{R_{e\,s}(v)}{\cosh(2(u-v))}$$

$$R_{e\,s}(u) = \frac{1}{1+e^{\epsilon(u)}} \exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dv \frac{\log(1+\eta_s^2(v))}{\cosh(2(u-v))}\right)$$

$$R_{o\,s}(u) = \frac{R_{e\,s}(u)}{\pi}\int_{-\infty}^{\infty} dv \frac{\arctan(\eta_s(v))}{\cosh(2(u-v))^2}$$

For $\mathcal{K}_+$:

$$\eta_+ = 4\,\mathrm{P.V.}\int_{-\infty}^{\infty} dv \frac{\coth(2v)\,R_{e\,+}(v)}{\cosh(2(u-v))}$$

$$R_{e\,+}(u) = \frac{\cosh(u)^2}{\cosh(2u)\,(1+e^{\epsilon(u)})} \exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dv \frac{\log(1+\eta_+^2(v))}{\cosh(2(u-v))}\right)$$

$$R_{o\,+}(u) = \frac{2\,R_{e\,+}(u)\coth(2u)}{\pi}\int_{-\infty}^{\infty} dv \frac{\arctan(\eta_+(v))}{\cosh(2(u-v))^2}$$

Solving TBA

$$\eta_s = 2\int_{-\infty}^{\infty} dv \frac{R_{\mathrm{e}\,s}(v)}{\cosh(2(u-v))}$$

$$R_{\mathrm{e}\,s}(u) = \frac{1}{1+e^{\epsilon(u)}}\exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dv \frac{\log(1+\eta_s^2(v))}{\cosh(2(u-v))}\right)$$

$$R_{\mathrm{o}\,s}(u) = \frac{R_{\mathrm{e}\,s}(u)}{\pi}\int_{-\infty}^{\infty} dv \frac{\arctan(\eta_s(v))}{\cosh(2(u-v))^2}$$

Expand in z: $\quad \eta = \sum_{k=1}^{\infty} \eta^{(k)} z^k \quad R_{\mathrm{e}} = \sum_{k=0}^{\infty} R_{\mathrm{e}}^{(2k)} z^{2k} \quad R_{\mathrm{o}} = \sum_{k=0}^{\infty} R_{\mathrm{o}}^{(2k+1)} z^{2k+1}$

Easy to solve iteratively: $\quad e^{-\epsilon} \to R_e^{(0)} \to \eta^{(1)} \to R_o^{(1)} \to R_e^{(2)} \to \ldots$

$$\mathrm{tr}\, K^{*2n+1} = \frac{1}{\pi^{2n+1}}\int du\, R_{\mathrm{e}}^{(2n)}(u)$$

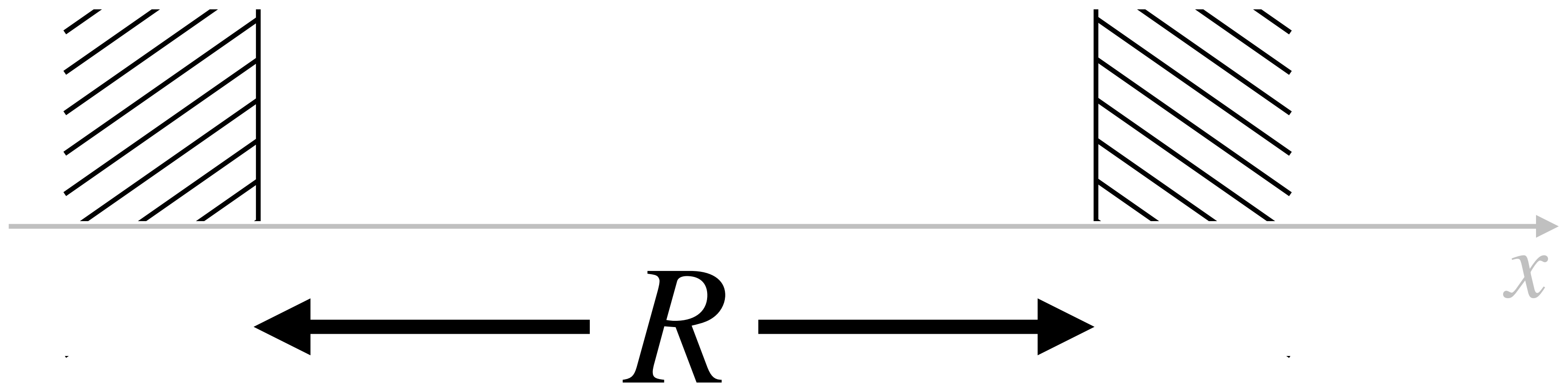
$$\mathrm{tr}\, K^{*2n} = \frac{1}{\pi^{2n}}\int du\, R_{\mathrm{o}}^{(2n-1)}(u)$$

Easy to compute arbitrarily high powers of n in contrast with multiple nested integrals.

Physical limits

CFT or UV limit of sinh-Gordon: Liouville CFT

Open string Lagrangian defined on a strip



$$L = \int_0^R dx \left(\frac{1}{4\pi}(\partial\phi)^2 + 2\mu \cosh(2b\phi) \right) + 2\mu_B \left(\cosh(b\,\phi)\,|_{x=0} + \cosh(b\,\phi)\,|_{x=R} \right)$$

Rescale width of the strip to 2π

$$L = \int_0^{2\pi} dx \left(\frac{1}{4\pi}(\partial\phi)^2 + \mu \left(\frac{R}{2\pi}\right)^{2+2b^2} (e^{2b\phi} + e^{-2b\phi}) \right)$$

$$+\mu_B \left(\frac{R}{2\pi}\right)^{1+b^2} \left((e^{b\phi} + e^{-b\phi})\,|_{x=0} + (e^{b\phi} + e^{-b\phi})\,|_{x=2\pi} \right)$$

UV limit $R \to 0$, neglect one of the exponentials: boundary Liouville

Closed string channel: $g-$function in UV $\leftrightarrow$ disc one point function in boundary Liouville

States in Liouville/Sinh-Gordon

In Liouville

Consider the classical limit $b \to 0$

Consider a field configuration constant in space $\phi(t, x) = \phi_0(t)$ (minisuperspace)

States characterized by canonical momentum P conjugate to ϕ_0

In Sinh-Gordon

Same approximation but $P = P(L)$

In the UV limit, ground state energy

$$E(L) = -\pi c_{\text{eff}}/6L \to \text{CFT beheviour with } c_{eff} \to c_L = 1 - 24P^2$$

Close to it, we take

$$c_{\text{eff}} = 1 - 24\, P(L)^2 + \mathcal{O}(L^2)$$

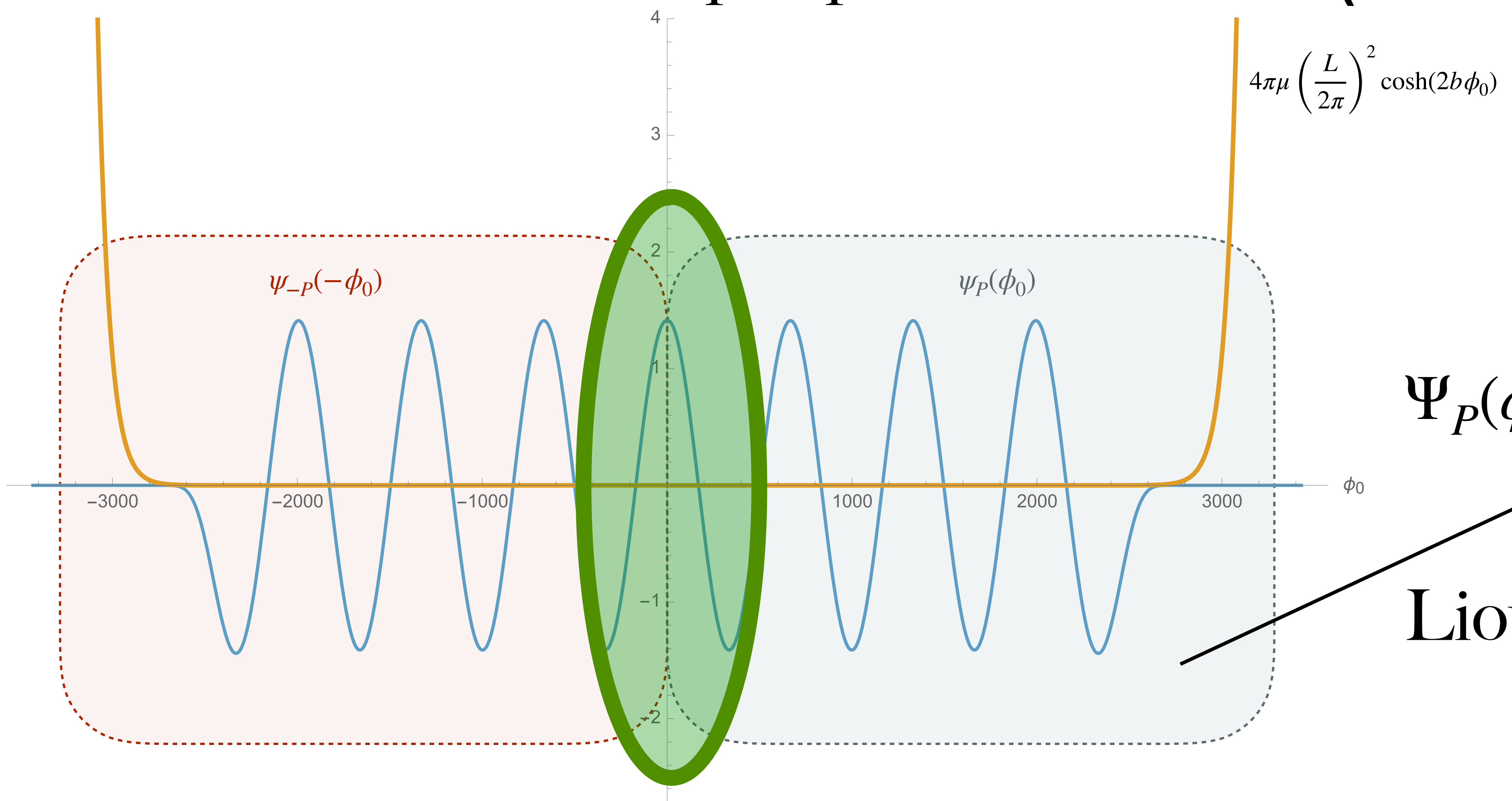
← define $P(L)$ in terms of c_{eff}

[Zamolodchikov, Zamolodchikov' 95]

Schrödinger equation

Sinh-Gordon ground-state wave-function in minisuperspace:

$$\left(-\frac{1}{2}\frac{d^2}{d\phi_0^2} + 4\pi\mu\left(\frac{L}{2\pi}\right)^2\cosh(2b\phi_0) - 2P^2\right)\Psi_P(\phi_0) = 0\,.$$



$$\Psi_P(\phi_0) \simeq \frac{2}{\Gamma(-2iP/b)}\left(\frac{\pi\mu L^2}{4\pi^2 b^2}\right)^{-iP/b} K_{2iP/b}\left(\frac{\sqrt{\pi\mu}L}{2\pi b}e^{b\phi_0}\right) \equiv \psi_P(\phi_0)$$

Liouville wave-function

Bessel K

Compatibility → quantization condition:

$$S_{\rm cl}(P)^2\left(\frac{L}{2\pi}\right)^{-8iP/b} = 1$$

Classical reflection matrix

$$P \sim -\frac{1}{\log(L/2\pi)} + \mathcal{O}\left(\frac{1}{\log(L/2\pi)^2}\right) \quad \text{as } L \to 0$$

Ground state of Sinh-Gordon in CFT limit ↔ Liouville $P = 0$ state

Liouville Boundary Data

Compare with one-point function of the boundary Liouville (also known as FZZT states)

$$\langle B_s | \psi_P \rangle_{\rm L} = \left(\pi\mu\gamma(b^2)\right)^{-iP/b} \Gamma(1+2ibP)\,\Gamma(1+2iP/b)\frac{\cos(2\pi P s)}{iP}$$

IR singularity!

Need to subtract the pole! Back to the classical limit:

$$\langle B_s | \Psi_P \rangle^{\rm cl}_{\rm shG} = \int_{-\infty}^{\infty} d\phi_0\, \Psi_P(\phi_0) \varphi_B(\phi_0)$$

Bulk wave-function

Boundary wave-function:

$$\varphi_B(\phi_0) = \exp\left(-2\,\mu_B \cosh(b\phi_0)\right)$$

Split integration regions, and approximate each region by Liouville:

$$\langle B_s | \Psi_P \rangle^{\rm cl}_{\rm shG} \simeq \langle B_s | \psi_{-P} \rangle^{\rm cl}_{\rm L} + \langle B_s | \psi_P \rangle^{\rm cl}_{\rm L} + \frac{i}{2P}\left(S_{\rm cl}(P) - \frac{1}{S_{\rm cl}(P)}\right) + \mathcal{O}(L)$$

Independent of boundary parameter

$$U_{s_1,s_0}(P) \equiv \frac{\langle B_{s_1} | \Psi_P \rangle_{\rm shG} - \langle B_{s_0} | \Psi_P \rangle_{\rm shG}}{\sqrt{\langle \Psi_P | \Psi_P \rangle}}$$

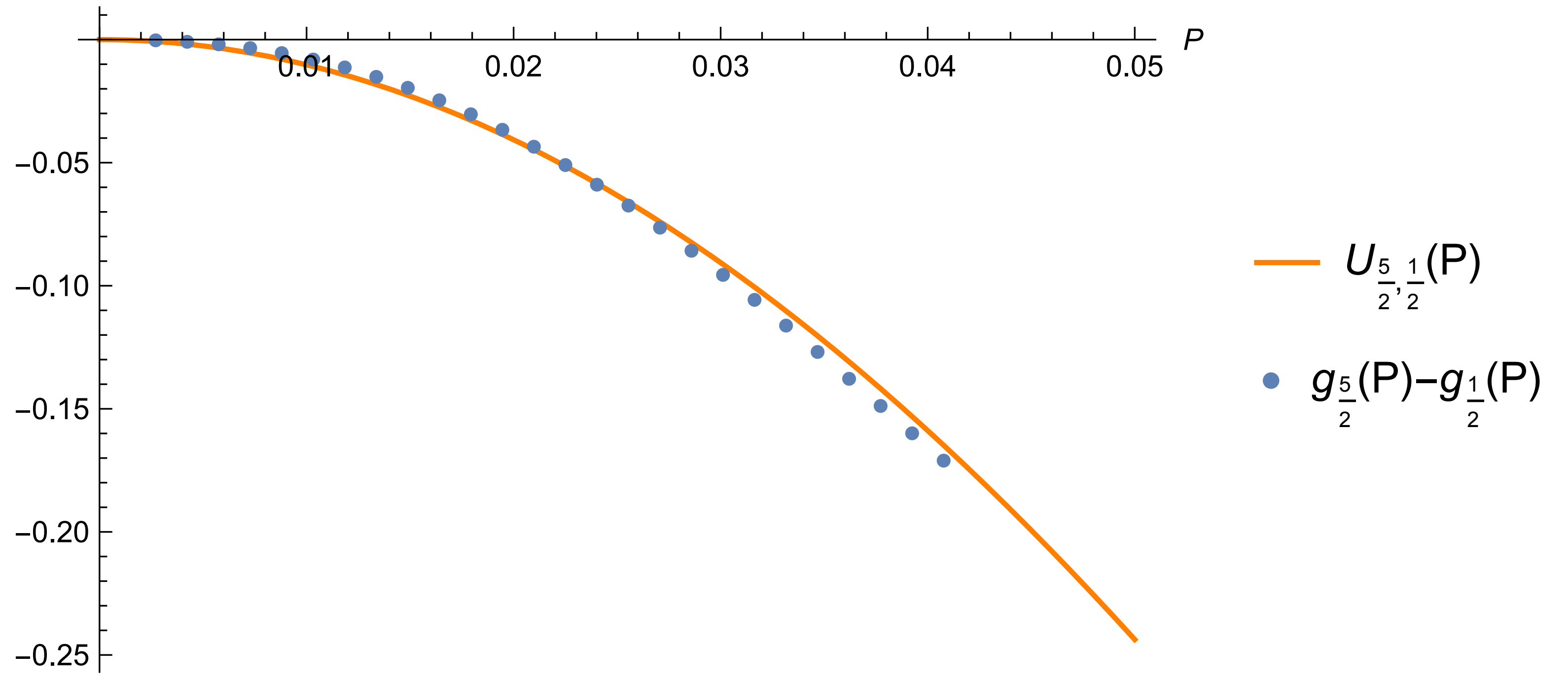
Reference one-point function

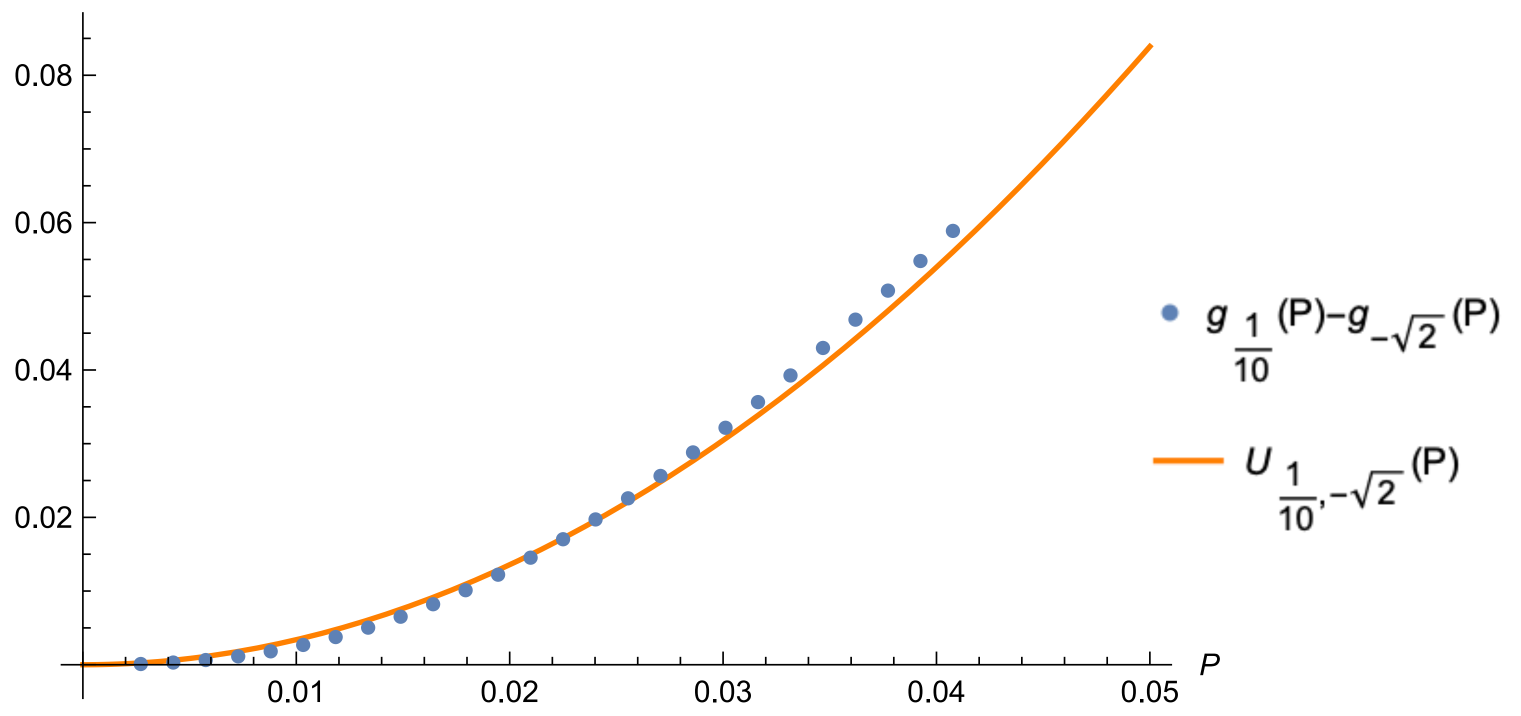
Comparison with Liouville

Solve Tracy-Widom TBA in the UV limit and compare with

$$U_{s_1,s_0}(P) = \frac{1}{\sqrt{\pi}}\left(\langle B_{s_1}|\psi_{-P}\rangle_{\mathrm{L}} + \langle B_{s_1}|\psi_P\rangle_{\mathrm{L}}\right) - \frac{1}{\sqrt{\pi}}\left(\langle B_{s_0}|\psi_{-P}\rangle_{\mathrm{L}} + \langle B_{s_0}|\psi_P\rangle_{\mathrm{L}}\right)$$

For any boundary parameters:





Should be useful to study excited states and test [Kostov, Serban, Vu '19; Jiang, Komatsu, Vescovi '20]

Separation of Variables

Lukyanov found a formula for one-point function in sinh-Gordon at finite volume. [Lukyanov' 01]

For the identity operator:

$$\langle \Omega | \Omega \rangle = \lim_{N \to \infty} \mathcal{I}_N$$

$$\mathcal{I}_N \equiv \frac{1}{(2N+1)!} \int_{-\infty}^{\infty} \prod_{k=-N}^{N} \frac{d\theta_k \left(Q(\theta_k)\right)^2}{2\pi} \prod_{-N \leq j < k \leq N} \Delta(\theta_j, \theta_k)$$

$$1 + e^{-\epsilon(u)} = Q^{++}(u) Q^{--}(u)$$

$$\Delta(\theta_j, \theta_k) \equiv \left(2 \sinh \nu(\theta_j - \theta_k)\right) \left(2 \sinh \tilde{\nu}(\theta_j - \theta_k)\right)$$

$$\nu \equiv 1 + b^2 \qquad \tilde{\nu} \equiv 1 + b^{-2}$$

From the Vandermonde determinant formula we can rewrite

$$\langle \Omega | \Omega \rangle = \lim_{N\to\infty} \det \left[M_{j,k} \right]_{-N\le j,k\le N} \qquad M_{j,k} = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} (Q(\theta))^2 e^{2(\nu k+\tilde{\nu} j)\theta}$$

For parity symmetric Q-function $Q(-\theta) = Q(\theta)$, determinant factorizes

$$\det M = \frac{1}{2} \det M^- \det M^+$$

$$(M^-)_{s,t} = 2\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} Q(\theta)Q(-\theta)\sinh(2\nu s\theta)\sinh(2\tilde{\nu} t\theta) \qquad (1 \le s,t \le N)$$

$$(M^+)_{s,t} = 2\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} Q(\theta)Q(-\theta)\cosh(2\nu s\theta)\cosh(2\tilde{\nu} t\theta) \qquad (0 \le s,t \le N)$$

Analogy: Gaudin norm for parity symmetric states $\det G = \det G^+ \det G^-$

$\sim \langle \mathrm{MPS} | \mathbf{u} \rangle$ [Buhl-Mortensen, de Leeuw, Kristjansen, Zarembo '15]

Conjecture: $\langle B | \Omega \rangle \propto \det M^-$

$$\frac{\sqrt{\mathrm{Det}(1-\hat{G})}}{\mathrm{Det}(1-\hat{G}_+)} = \lim_{N\to\infty} \mathcal{N} \times \frac{\det M^-}{\sqrt{\det M}} = \lim_{N\to\infty} \mathcal{N} \times \frac{\overline{\mathcal{I}}_N}{\sqrt{\mathcal{I}_N}}$$

$$\overline{\mathcal{I}}_N = \frac{1}{N!}\int_{-\infty}^{\infty} \left(\prod_{k=1}^{N} \frac{d\theta_k \sinh(2\nu\theta_k)\sinh(2\tilde{\nu}\theta_k)Q(\theta_k)Q(-\theta_k)}{\pi}\right) \prod_{1\le j,k\le N} \overline{\Delta}(\theta_j,\theta_k)$$

with

$$\overline{\Delta}(\theta_j,\theta_k) \equiv \left[2\cosh(2\nu\theta_j) - 2\cosh(2\nu\theta_k)\right]\left[2\cosh(2\tilde{\nu}\theta_j) - 2\cosh(2\tilde{\nu}\theta_k)\right]$$

$$= \left(\sinh^2(\nu\theta_j) - \sinh^2(\nu\theta_k)\right)\left(\sinh^2(\tilde{\nu}\theta_j) - \sinh^2(\tilde{\nu}\theta_k)\right)$$

- Selection rule: $\det M^-$ vanishes if the Q-function is not parity-symmetric, $Q(\theta) \neq Q(-\theta)$.
  (Boundary state is annihilated by the action of odd conserved charger under parity)
- Still need to fix $\mathcal{N}$
- Same trick works in XXX spin-chain: from the norm one can get SoV representation for $\langle \text{Néel} | \mathbf{u} \rangle$
- Does this trick works for higher-rank cases?

Future directions

- Extend to more general types of kernels and theories with bound-states/internal degrees of freedom.
- $\mathcal{N} = 4$ SYM g-function.
- Physical interpretation of the equations
- Analytically solution of these equations in UV/ IR?
- Excited States? Dorey-Tateo analytic continuation for Tracy-Widom TBA? Use Liouville to test.
- Sharpen/improve SoV conjecture. Guess higher-rank overlaps from norms?
- Applications in the computation of S^3 partition function of superconformal Chern-Simons with OSp gauge groups, where $\mathcal{K}_+$ appear.

Thank you