# Completeness of Bethe equations

Based on **2004.02865** with D.Chernyak & S.Leurent


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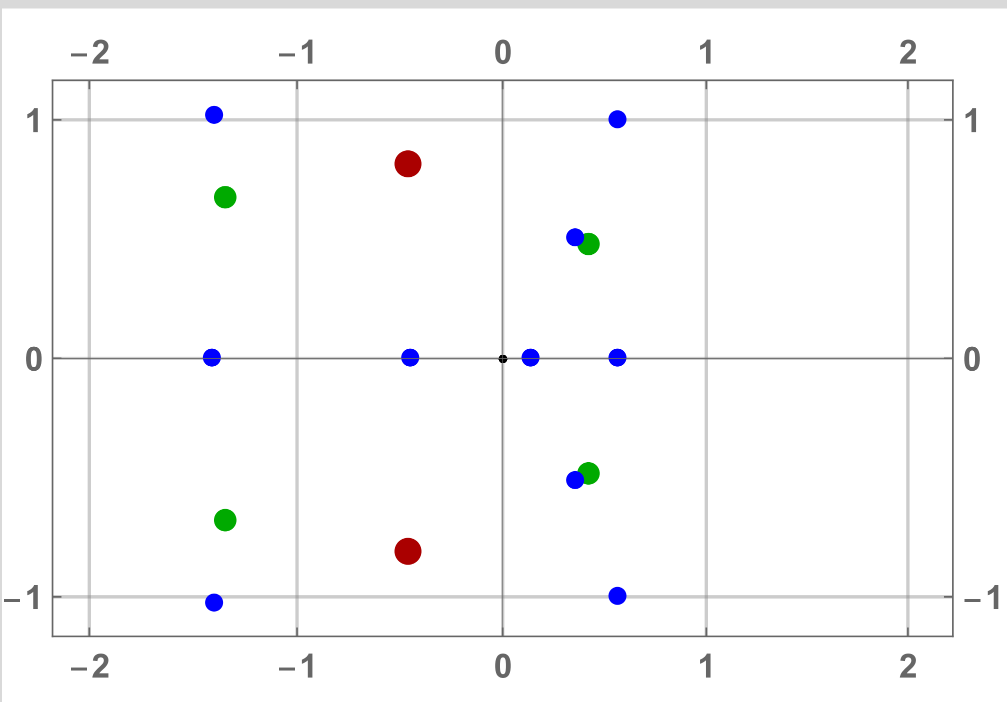
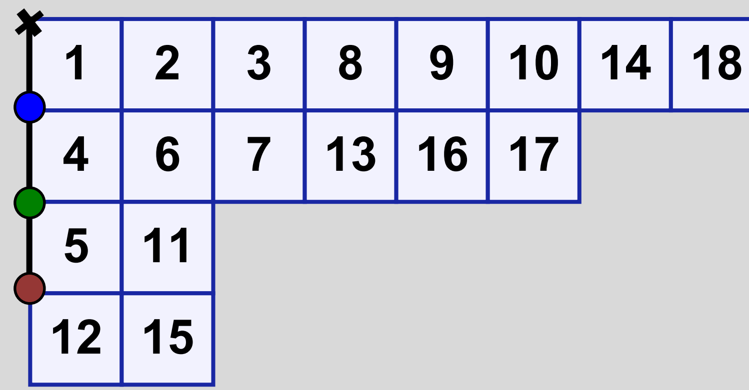
*London integrability literature club, 07/05/2020*



# Part 0
# Statement of results

# Statements

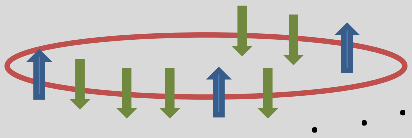
- **Completeness**: Algebraic number of solutions of good Bethe equations is always the right one

- **Faithfulness**: There is a one-to-one correspondence between distinct solutions and eigenstates of the Bethe algebra (a special set of commuting charges)

- **Simplicity**: If the Bethe algebra is diagonalisable then the joint spectrum of its commuting charges is simple (i.e. non-degenerate)

- **Explicit parameterisation:** There is an unambiguous way to label solutions with standard Young tableaux (which apparently gives a precise and rigorous definition of Bethe strings)

- **Completeness**: Algebraic number of solutions of good Bethe equations is always the right one



**BAE for gl(2) case:**

$$\xcancel{\prod_{\alpha=1}^{L}\frac{u_k-\theta_\alpha+\frac{\hbar}{2}}{u_k-\theta_\alpha-\frac{\hbar}{2}}=\prod_{j\neq k}^{M}\frac{u_k-u_j+\hbar}{u_k-u_j-\hbar}}$$

$$Q(u)=\prod_{i=1}^{M}(u-u_i)$$

**Wronskian Bethe equations:**

$$\begin{vmatrix} Q(u+\hbar/2) & Q(u-\hbar/2) \\ \tilde{Q}(u+\hbar/2) & \tilde{Q}(u-\hbar/2) \end{vmatrix} \propto \prod_{\alpha=1}^{L}(u-\theta_\alpha)$$

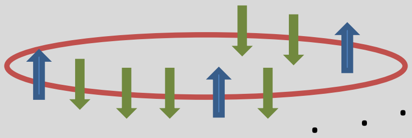
$$Q = u^M + \boldsymbol{c_1} u^{M-1} + \cdots + \boldsymbol{c_M}$$

$$\tilde{Q} = u^{L-M+1} + \tilde{\mathbf{c}}_{\mathbf{1}} u^{L-M} + \cdots + \tilde{\mathbf{c}}_{\boldsymbol{L-2M+1}} u^M + \cdots + \tilde{\mathbf{c}}_{\boldsymbol{L-M+1}}$$

$$\prod_{\alpha=1}^{L}(u-\theta_\alpha)=\sum_{\alpha=0}^{L}(-1)^\alpha \chi_\alpha\, u^{L-\alpha}$$

$$\boxed{W_\alpha[c]=\chi_\alpha}$$

Algebraic number of solutions is $\binom{L}{M}-\binom{L}{M-1}$



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**Wronskian Bethe equations:**

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# Statements are proven for

Spin chains in the defining representation of $\mathrm{gl}(m|n)$, both for periodic boundary conditions (no twist) and non-degenerate twist cases

- gl(2) – **[Mukhin, Tarasov, Varchenko ’07]** 0706.0688 [math.QA]

*“Bethe algebra of homogeneous XXX Heisenberg model has simple spectrum,”*

- gl(n) – **[Mukhin, Tarasov, Varchenko ’13]** 1303.1578 [math.AG]

“Spaces of quasi-exponentials and representations of the Yangian Y($gl_N$)”

- gl(1|1) – **[Huang, Lu, Mukhin ’18]** 1811.11225 [math.QA]

*”Solutions of $\mathfrak{gl}$m|n XXX Bethe ansatz equation and rational difference operators”*

**[Lu, Mukhin ’19]** 1910.13360 [math.QA]

*”On the supersymmetric XXX spin chains associated to $\mathfrak{gl}_{1|1}$”*

- gl(m|n) – **[Chernyak, Leurent, D.V. ’20]** 2004.02865 [math-ph]

*”Completeness of Wronskian Bethe equations for rational gl(m|n) spin chain”*

**Remark:** we know recipe for ”good Bethe equations” for spin chains in any integer-weight unitary representations of su(p,q|m), it is a direct generalisation of **[Marboe, D.V. ’18]**

This talk is only about periodic boundary conditions (actually more complicated, but more fun)

# Part -1.a
# Introduction to polynomial rings

$\boxed{W_\alpha[c] - \chi_\alpha = 0}$ Q: how to count number of solutions?

$P_a(x_1, x_2, \dots, x_n) = 0$ a=1,…,m

$$\mathcal{W} = \frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{\langle P_1, P_2, \dots, P_m \rangle}$$

$\mathbb{C}[x_1, x_2, \dots, x_n]$ - ring of polynomials in $n$ variables, it is in particular a vector space spanned by monomials $x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}$.

$\langle P_1, P_2, \dots, P_m \rangle$ - ideal in $\mathbb{C}[x_1, x_2, \dots, x_n]$ generated by $P_a$ - collection of all polynomials that can be represented as $\sum_a q_a(x) P_a(x)$

Thm: # of solutions = dim $\mathcal{W}$

Example: $$\mathcal{W} = \frac{\mathbb{C}[x]}{\langle x^2 - a\, x + b \rangle}$$

$(1, x)$ form a basis in $\mathcal{W}$

$$\mathcal{W} = \frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{\langle P_1, P_2, \dots, P_m \rangle}$$

Thm: # of solutions = dim $\mathcal{W}$

$$\begin{cases} x(x\,y - 1) = 0 \\ y^2 - x\,y - 1 = 0 \end{cases} \longrightarrow \quad x = 0 \longrightarrow y^2 = 1 \longrightarrow y = \pm 1$$

or

$$x\,y = 1 \longrightarrow y^2 = 2 \longrightarrow y = \pm\sqrt{2}$$

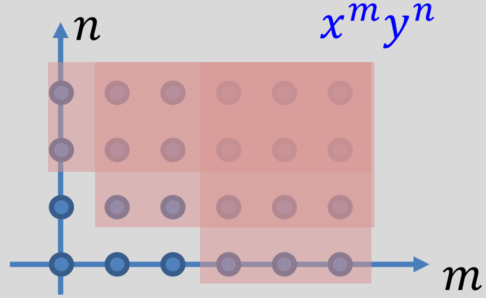
4 solutions

Gröbner basis in lex order $y \succ x$

$$\begin{cases} y^2 - 2\,x^2 - 1 \\ x\,y - 2\,x^2 \\ x^3 - \frac{1}{2}x \end{cases}$$

$\mathbb{C}[x, y]$

$x^m y^n$



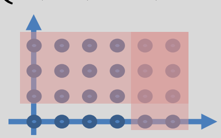

$(1, x, x^2, y)$ form a basis in $\mathcal{W}$

Parametric dependence

$$\begin{cases} x(x\,y - 1) - \chi_1 \\ y^2 - x\,y - \chi_2 \end{cases}$$

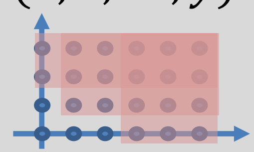
$\mathbb{C}(\chi_1, \chi_2)[x, y]$

$(1, x, x^2, x^3)$



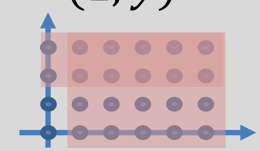
$\mathbb{C}(\chi_2)[x, y]$

$(1, x, x^2, y)$



$\mathbb{C}[x, y]$

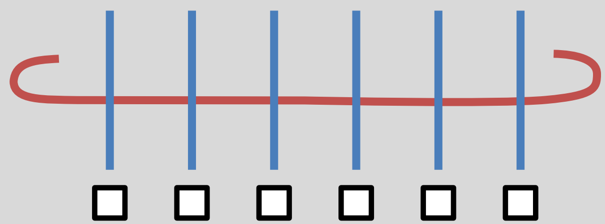
$(1, y)$



GB over $\mathbb{C}(\chi)$:

$$\begin{cases} y + \dfrac{(x^3 - x)(1 + \chi_2) - x^2 \chi_1 \chi_2 - \chi_1}{\chi_1{}^2} \\ x^4 + \dfrac{(x^3 - 2x)\chi_1 - x^2 - \chi_1{}^2}{1 + \chi_2} \end{cases} \xrightarrow{\chi_1 = 0} \begin{cases} y^2 - x^2(1 + \chi_2) - \chi_2 \\ x\,y - x^2(1 + \chi_2) \\ x^3 - \dfrac{x}{1 + \chi_2} \end{cases} \xrightarrow{\chi_2 = -1} \begin{cases} y^2 + 1 \\ x \end{cases}$$

# Part I
# Completeness

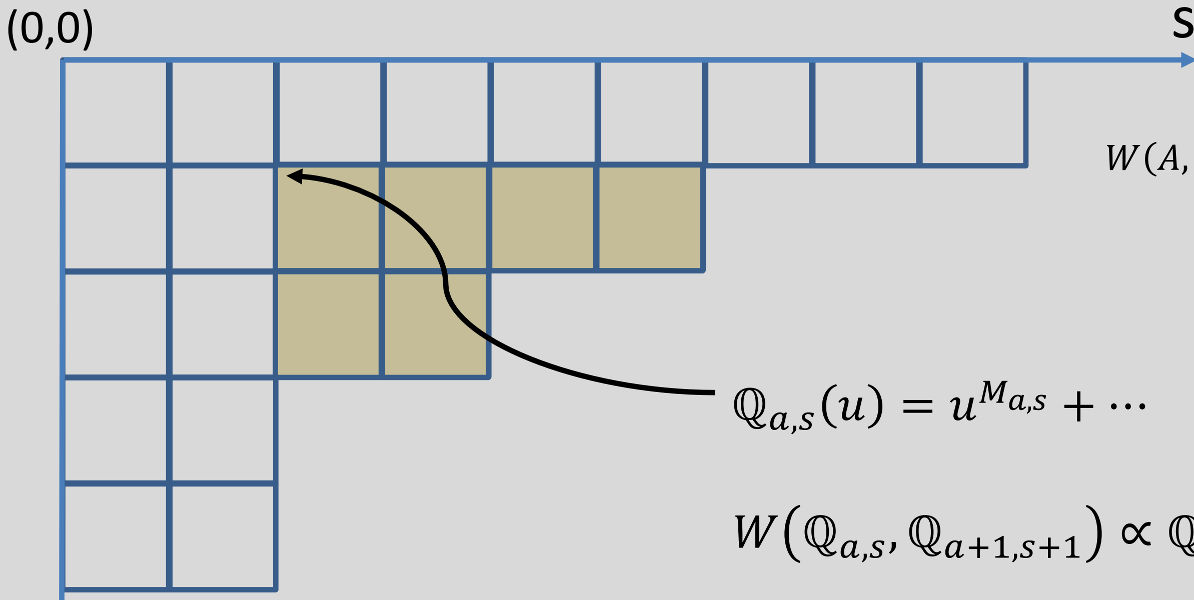


$\mathfrak{gl}(\mathrm{m}|\mathrm{n}) \times \mathbb{C}[S_L]$

- Restict to HWS of irreps with fixed Young diagram
- Dimension of this space = # of standard Young Tableaux

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 5 | 6 | 9 |
| 3 | 7 | 10 | | |
| 4 | 8 | | | |

# Q-system on Young diagram: definition

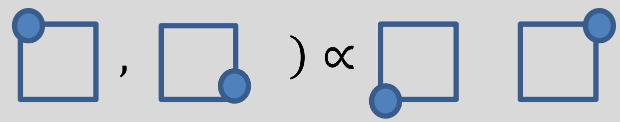



$$W(A,B) := \begin{vmatrix} A(u+\hbar/2) & A(u-\hbar/2) \\ B(u+\hbar/2) & B(u-\hbar/2) \end{vmatrix}$$

$\mathbb{Q}_{a,s}(u) = u^{M_{a,s}} + \cdots \qquad M_{a,s} = \#\text{ SE boxes}$

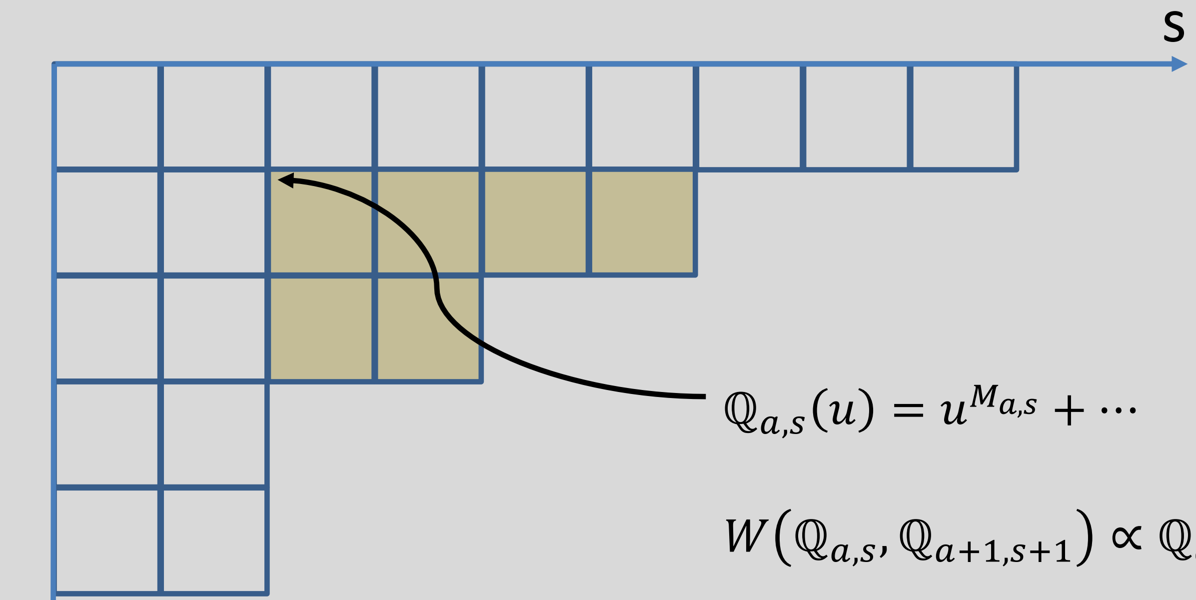
$$W\left(\mathbb{Q}_{a,s}, \mathbb{Q}_{a+1,s+1}\right) \propto \mathbb{Q}_{a+1,s}\mathbb{Q}_{a,s+1}$$

$W($ , $) \propto$



Quantisation condition: $\mathbb{Q}_{0,0} = \prod_{\alpha=1}^{L}(u-\theta_\alpha)$

# Q-system on Young diagram: parameterisations

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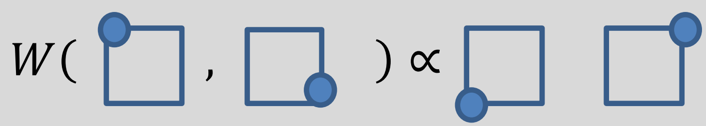



Quantisation condition: $\mathbb{Q}_{0,0} = \prod_{\alpha=1}^{L}(u-\theta_\alpha)$

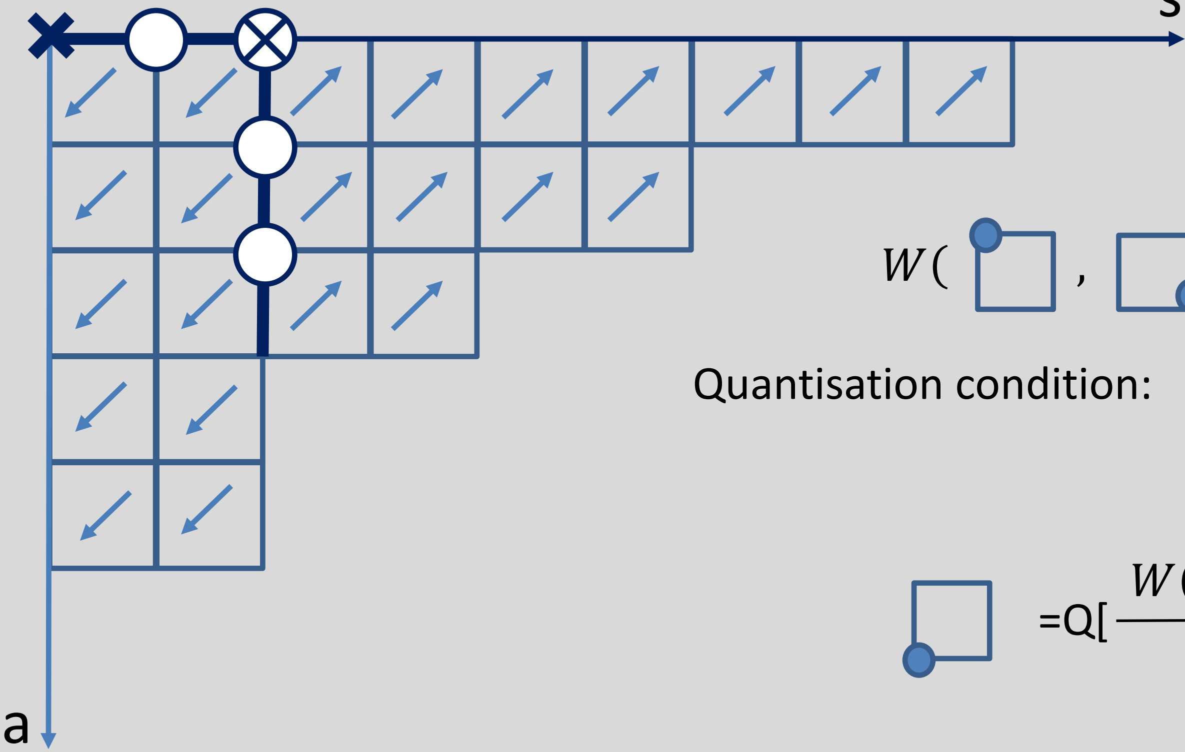
Parameterisation N1: Lazy – all coefficients subjected to all equations

$$\mathbb{Q}_{a,s} = u^{M_{a,s}} + \boldsymbol{c}_{\boldsymbol{a,s}}^{(\mathbf{1})} u^{M_{a,s}-1} + \cdots + \boldsymbol{c}_{\boldsymbol{a,s}}^{(M_{a,s})}$$

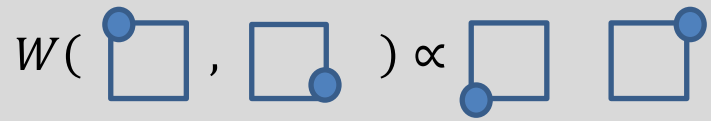
- Used in numerics

## Q-system on Young diagram: parameterisations

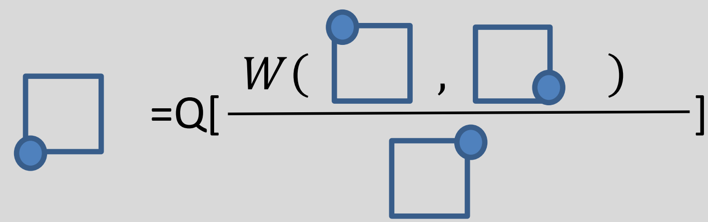
## N2: along Kac-Dynkin path




$$\frac{A}{B} = Q + \frac{R}{B}$$



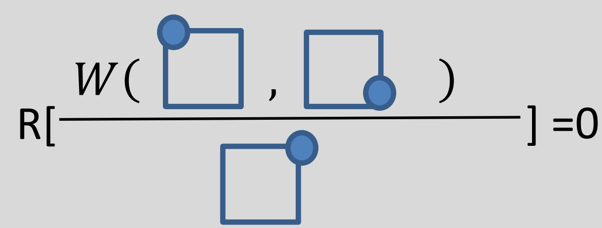

Quantisation condition: $\mathbb{Q}_{0,0} = \prod_{\alpha=1}^{L}(u - \theta_\alpha)$




## Parameterisation N2: along Kac-Dynkin path

$$\mathbb{Q}_{a,s} = u^{M_{a,s}} + \boldsymbol{c_{a,s}^{(1)}} u^{M_{a,s}-1} + \cdots + \boldsymbol{c_{a,s}^{(M_{a,s})}}$$

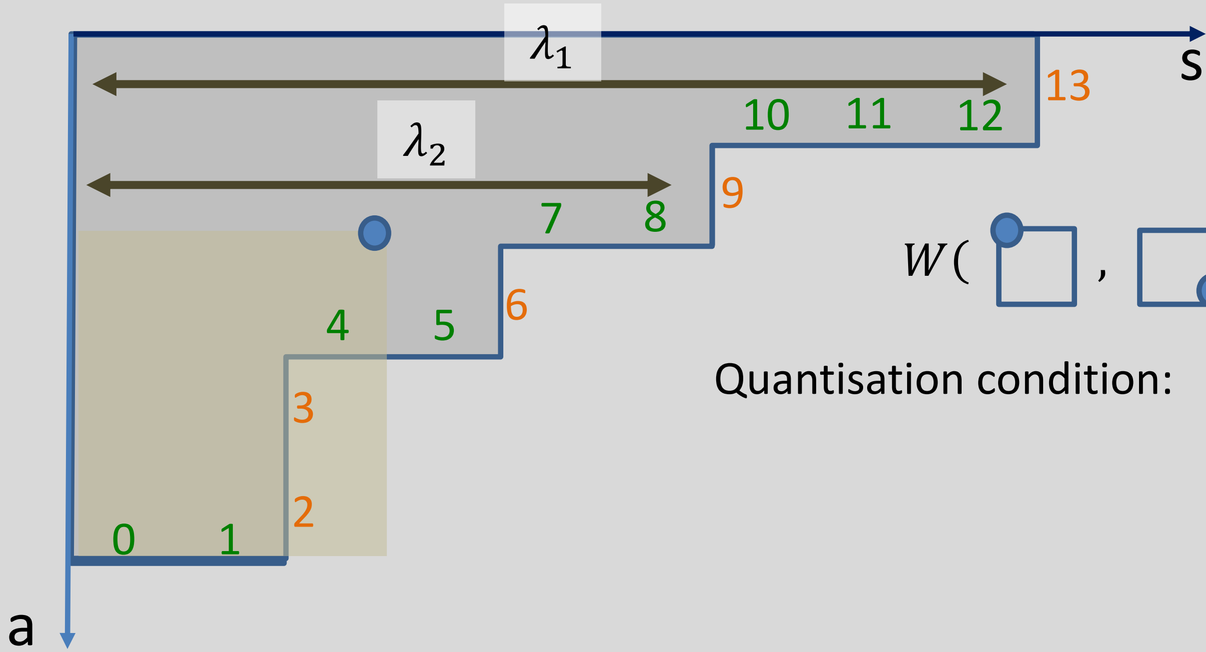
Good Bethe equations are ZRC:



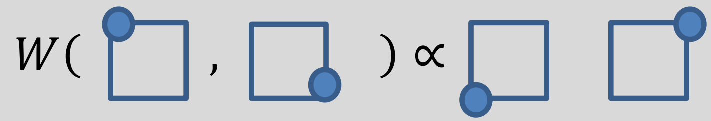

- Zeros of $\mathbb{Q}_{a,s}(u)$ on the path are Bethe roots of nested Bethe equations

(cf. [Kazakov, Sorin, Zabrodin '07])

## N3: by Bosonised Q-functions




$\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{h_\Lambda})$




Quantisation condition: $\mathbb{Q}_{0,0} = \prod_{\alpha=1}^{L}(u-\theta_\alpha)$

$Q_a = u^{\hat{\lambda}_a} + \cdots$ $\qquad \hat{\lambda}_a = \lambda_a + h_\Lambda - a$ $\qquad Q_a = u^{\hat{\lambda}_a} + \boldsymbol{c}_{\boldsymbol{a}}^{(\mathbf{1})} u^{\hat{\lambda}_a - 1} + \cdots + \boldsymbol{c}_{\boldsymbol{a}}^{(\hat{\lambda}_a - \hat{\lambda}_b)} u^{\hat{\lambda}_b} + \ldots$

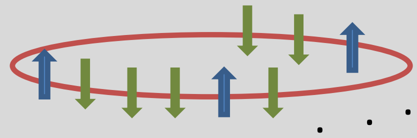
$B_s = u^{\bar{\boldsymbol{\nu}}_{\boldsymbol{s}}} + \cdots$ $\qquad \bar{\nu}_s = \lambda'_1 - \lambda'_s + s - 1$ $\qquad B_s := u^{\bar{\boldsymbol{\nu}}_{\boldsymbol{s}}}$

$$\boxed{\mathbb{Q}_{a,s} = W(Q_{a+1}, Q_{a+2}, \ldots, Q_{h_\Lambda}, B_1, B_2, \ldots, B_s)}$$

consequence of bosonisation trick in [Gromov, Kazakov, Leurent, Tsuboi '10]

$$\boxed{\mathbb{Q}_{0,0} \propto W(Q_1, Q_{a+2}, \ldots, Q_{h_\Lambda})}$$

Wronskian Bethe equations for bosonic SU($h_\Lambda$) system



**BAE for gl(2) case:**

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$$Q(u)=\prod_{i=1}^{M}(u-u_i)$$

**Wronskian Bethe equations:**

$$\begin{vmatrix} Q(u+\hbar/2) & Q(u-\hbar/2) \\ \tilde{Q}(u+\hbar/2) & \tilde{Q}(u-\hbar/2) \end{vmatrix} \propto \prod_{\alpha=1}^{L}(u-\theta_\alpha)$$

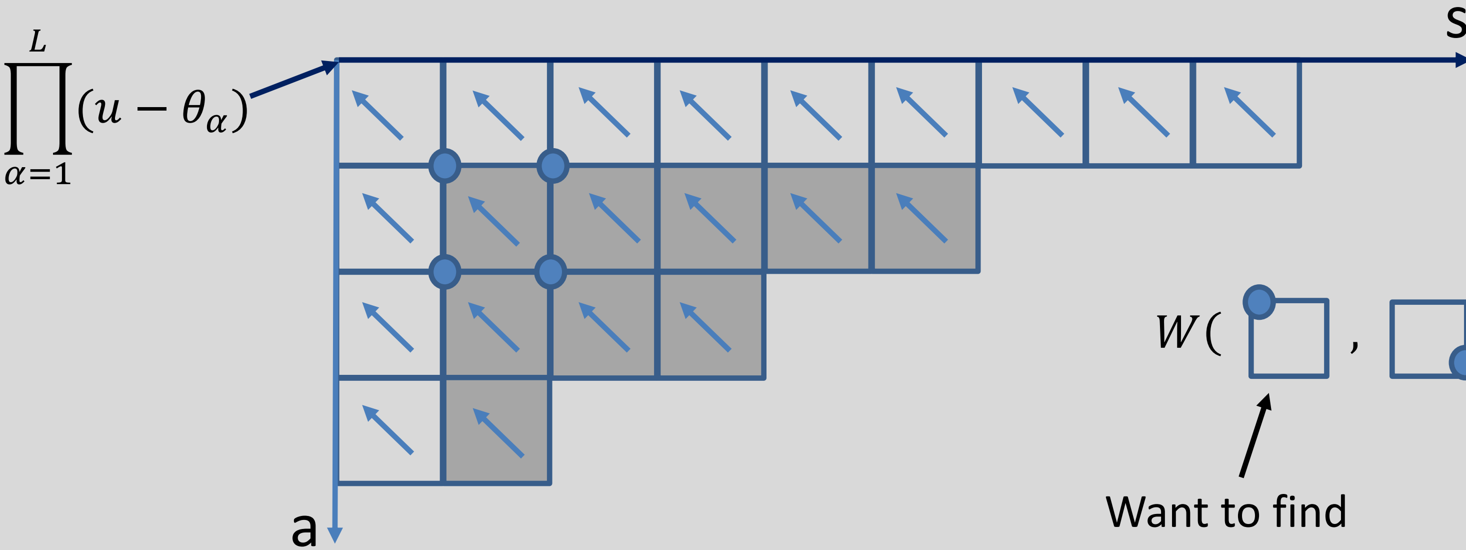
$$Q = u^M + \boldsymbol{c_1} u^{M-1} + \cdots + \boldsymbol{c_M}$$

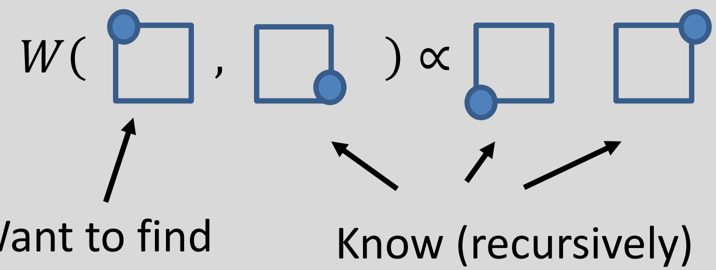
$$\tilde{Q} = u^{L-M+1} + \tilde{\mathbf{c}}_{\mathbf{1}} u^{L-M} + \cdots + \tilde{\mathbf{c}}_{\boldsymbol{L-2M+1}} u^M + \cdots + \tilde{\mathbf{c}}_{\boldsymbol{L-M+1}}$$

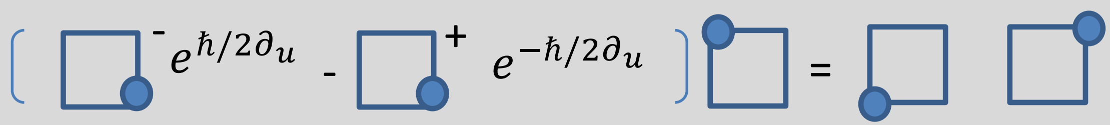
$$\prod_{\alpha=1}^{L}(u-\theta_\alpha)=\sum_{\alpha=0}^{L}(-1)^\alpha \chi_\alpha u^{L-\alpha}$$

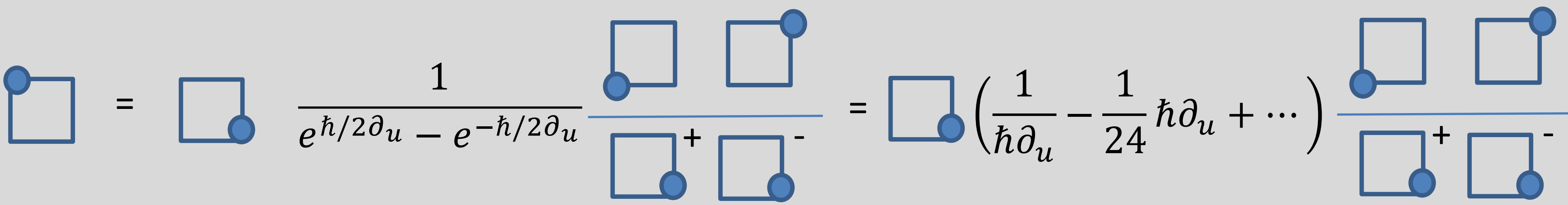
$$\boxed{W_\alpha[c]=\chi_\alpha}$$

Algebraic number of solutions is $\binom{L}{M}-\binom{L}{M-1}$




$$\mathbb{Q}_{a,s} = \mathbb{Q}^0{}_{a,s} + \boldsymbol{c_{a+1,s+1}}\mathbb{Q}_{a+1,s+1}$$

$$\frac{1}{\partial_u}u^p = \frac{u^{p+1}}{p+1} + const$$

$$\mathrm{YW}[c] = \prod_{\alpha=1}^{L}(u-\theta_\alpha)$$

$$\boxed{YW_\alpha[c] = \chi_\alpha}$$

Wronskian Bethe equations that we shall use

WBE: $\boxed{YW_\alpha[c] = \chi_\alpha}$

$$\mathbb{Q}_{a,s} = \mathbb{Q}^0{}_{a,s} + \boldsymbol{c}_{\boldsymbol{a+1,s+1}}\mathbb{Q}_{a+1,s+1}$$

$$\begin{array}{|c|c|c|c|}
\hline
\boldsymbol{c}_{\boldsymbol{1,1}} & \boldsymbol{c}_{\boldsymbol{1,2}} & \boldsymbol{c}_{\boldsymbol{1,3}} & \boldsymbol{c}_{\boldsymbol{1,4}} \\
\hline
\boldsymbol{c}_{\boldsymbol{2,1}} & \boldsymbol{c}_{\boldsymbol{2,2}} & \multicolumn{2}{c}{} \\
\cline{1-2}
\boldsymbol{c}_{\boldsymbol{3,1}} & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array}$$

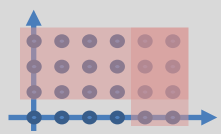
$$\mathcal{W} = \frac{\mathbb{C}(\chi)[c]}{\langle YW_\alpha[c] - \chi_\alpha \rangle}$$

Parametric dependence

$$\begin{cases} x(x\,y - 1) - \chi_1 \\ y^2 - x\,y - \chi_2 \end{cases}$$

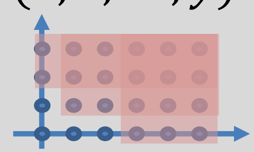
$\mathbb{C}(\chi_1, \chi_2)[x, y]$

$(1, x, x^2, x^3)$



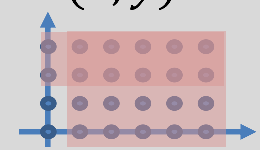
$\mathbb{C}(\chi_2)[x, y]$

$(1, x, x^2, y)$



$\mathbb{C}[x, y]$

$(1, y)$



GB over $\mathbb{C}(\chi)$:

$$\begin{cases} y + \dfrac{(x^3 - x)(1+\chi_2) - x^2\chi_1\chi_2 - \chi_1}{\chi_1{}^2} \\[2ex] x^4 + \dfrac{(x^3 - 2x)\chi_1 - x^2 - \chi_1{}^2}{1+\chi_2} \end{cases} \xrightarrow{\chi_1 = 0} \begin{cases} y^2 - x^2(1+\chi_2) - \chi_2 \\ x\,y - x^2(1+\chi_2) \\ x^3 - \dfrac{x}{1+\chi_2} \end{cases} \xrightarrow{\chi_2 = -1} \begin{cases} y^2 + 1 \\ x \end{cases}$$

WBE: $\boxed{YW_\alpha[c] = \chi_\alpha}$

$$\mathbb{Q}_{a,s}(u) = \mathbb{Q}^0{}_{a,s}(u) + \boldsymbol{c_{a+1,s+1}}\mathbb{Q}_{a+1,s+1}(u)$$

| | | | |
|---|---|---|---|
| $\boldsymbol{c_{1,1}}$ | $\boldsymbol{c_{1,2}}$ | $\boldsymbol{c_{1,3}}$ | $\boldsymbol{c_{1,4}}$ |
| $\boldsymbol{c_{2,1}}$ | $\boldsymbol{c_{2,2}}$ | | |
| $\boldsymbol{c_{3,1}}$ | | | |

$$\mathcal{W} = \frac{\mathbb{C}(\chi)[c]}{\langle YW_\alpha[c] - \chi_\alpha \rangle}$$

**Thm: Wronskian algebra is a free $\mathbb{C}[\chi]$ - module**

Wronskian algebra: $$\mathcal{W} = \frac{\mathbb{C}[\chi][c]}{\langle YW_\alpha[c] - \chi_\alpha \rangle}$$

Exists a basis $\mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_d$ such that any element $X$ of $\mathcal{W}$ is represented as

$$X = r_1(\chi)\mathfrak{b}_1 + r_2(\chi)\mathfrak{b}_2 + \cdots + r_d(\chi)\mathfrak{b}_d$$ for some polynomials $r_a(\chi)$

Key elements of the proof: to prove that all $c$ are bounded if $\chi$ are finite and reduce the proof to the case of Quillen-Suslin theorem.

WBE: $\boxed{YW_\alpha[c] = \chi_\alpha}$

$$\mathbb{Q}_{a,s}(u) = \mathbb{Q}^0{}_{a,s}(u) + \boldsymbol{c}_{a+1,s+1}\mathbb{Q}_{a+1,s+1}(u)$$

$$\begin{array}{|c|c|c|c|}
\hline
\boldsymbol{c}_{1,1} & \boldsymbol{c}_{1,2} & \boldsymbol{c}_{1,3} & \boldsymbol{c}_{1,4} \\
\hline
\boldsymbol{c}_{2,1} & \boldsymbol{c}_{2,2} & \multicolumn{2}{c}{} \\
\cline{1-2}
\boldsymbol{c}_{3,1} & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array}$$

$$\mathcal{W} = \frac{\mathbb{C}[\chi][c]}{\langle YW_\alpha[c] - \chi_\alpha \rangle}$$

$$X = r_1(\chi)\mathfrak{b}_1 + r_2(\chi)\mathfrak{b}_2 + \cdots + r_d(\chi)\mathfrak{b}_d$$

**# of solutions = $\mathbf{dim}_{\mathbb{C}[\chi]}\mathcal{W} = d$ via Hilbert series**

(Hilbert series = character = index = partition function)

$$\deg \boldsymbol{c}_{a,s} = \boldsymbol{h}_{a,s} \qquad \deg \chi_\alpha = \alpha$$

$$\mathbb{Q}_{0,0} = \prod_{\alpha=1}^{L}(u - \theta_\alpha) = \sum_{\alpha=0}^{L}(-1)^\alpha \chi_\alpha\, u^{L-\alpha}$$

$$\mathcal{W}_0 \subset \mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots$$

$$\mathcal{W} \cong \mathbb{C}[c]$$

$$\mathcal{W}_k \mathcal{W}_{k'} \subset \mathcal{W}_{k+k'}$$

$$ch_{\mathcal{W}}(t) = \prod_{a,s} \frac{1}{1 - t^{\boldsymbol{h}_{a,s}}}$$

$$ch_{\mathcal{W}}(t) = \sum_{k=0}^{\infty} \dim(\mathcal{W}_k/\mathcal{W}_{k-1})t^k$$

WBE: $\boxed{YW_\alpha[c] = \chi_\alpha}$

$$\mathbb{Q}_{a,s}(u) = \mathbb{Q}^0{}_{a,s}(u) + \boldsymbol{c_{a+1,s+1}}\mathbb{Q}_{a+1,s+1}(u)$$

$$\begin{array}{|c|c|c|c|}
\hline
\boldsymbol{c_{1,1}} & \boldsymbol{c_{1,2}} & \boldsymbol{c_{1,3}} & \boldsymbol{c_{1,4}} \\
\hline
\boldsymbol{c_{2,1}} & \boldsymbol{c_{2,2}} & & \\
\cline{1-2}
\boldsymbol{c_{3,1}} & & & \\
\cline{1-1}
\end{array}$$

$$\mathcal{W} = \frac{\mathbb{C}[\chi][c]}{\langle YW_\alpha[c] - \chi_\alpha \rangle}$$

$$X = r_1(\chi)\mathfrak{b}_1 + r_2(\chi)\mathfrak{b}_2 + \cdots + r_d(\chi)\mathfrak{b}_d$$

**# of solutions = $\mathbf{dim}_{\mathbb{C}[\chi]}\mathcal{W} = d$ via Hilbert series**

(Hilbert series = character = index = partition function)

$\deg \boldsymbol{c_{a,s}} = \boldsymbol{h_{a,s}}$ $\quad$ $\deg \chi_\alpha = \alpha$

$$\mathbb{Q}_{0,0} = \prod_{\alpha=1}^{L}(u - \theta_\alpha) = \sum_{\alpha=0}^{L}(-1)^\alpha \chi_\alpha\, u^{L-\alpha}$$

$$ch_{\mathcal{W}}(t) = \prod_{a,s}\frac{1}{1 - t^{\boldsymbol{h_{a,s}}}}$$

$$ch_{\mathbb{C}[\chi]}(t) = \prod_{\alpha=1}^{L}\frac{1}{1 - t^\alpha}$$

$$\mathbf{dim}_{\mathbb{C}[\chi]}\mathcal{W} = \frac{ch_{\mathcal{W}}(t)}{ch_{\mathbb{C}[\chi]}(t)}\,|t \to 1 = \frac{L!}{\prod_{a,s}\ \boldsymbol{h_{a,s}}}$$

Kostka-Foulkes polynomial $\qquad$ Hook formula!

# Summary

- Nested Bethe equations should be replaced by Wronskian Bethe equations

$$\prod_{\ell=1}^{L}\frac{u_i^{(k)}-\theta_\ell+\frac{\hbar}{2}\delta_{k,1}}{u_i^{(k)}-\theta_\ell-\frac{\hbar}{2}\delta_{k,1}} = (-1)^{\frac{c_{kk}}{2}}\prod_{k'=1}^{3}\prod_{i'=1}^{M_{k'}}\frac{u_i^{(k)}-u_{i'}^{(k')}+\hbar c_{kk'}}{u_i^{(k)}-u_{i'}^{(k')}-\hbar c_{kk'}} \longrightarrow \boxed{YW_\alpha[c]=\chi_\alpha}$$

- WBE depend **only on Young diagram** but not the rank of the gl(m|n) algebra

- We introduced WBE using YD-natural parameterisation

| | | | |
|---|---|---|---|
| $\cancel{c_{1,1}}$ | $\cancel{c_{1,2}}$ | $\cancel{c_{1,3}}$ | $\cancel{c_{1,4}}$ |
| $\cancel{c_{2,1}}$ | $\cancel{c_{2,2}}$ | | |
| $\cancel{c_{3,1}}$ | | | |

- Wronskian algebra $\quad \mathcal{W}=\dfrac{\mathbb{C}[\chi][c]}{\langle YW_\alpha[c]-\chi_\alpha\rangle}$

- Wronskian algebra is a free $\mathbb{C}[\chi]$-module –> number of solutions is the same for any numerical $\bar{\chi}$

- Computed number of solutions (with multiplicities) using Hilbert series

$$\mathbf{dim}_{\mathbb{C}[\chi]}\mathcal{W}=\frac{ch_{\mathcal{W}}(t)}{ch_{\mathbb{C}[\chi]}(t)}\,|t\to 1=\frac{L!}{\prod_{a,s}\ \boldsymbol{h_{a,s}}}$$

# Part II
# Faithfulness

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Nil}(\mathcal{W}_\Lambda(\bar{\theta})) \arrow[r] \arrow[d, "\varphi_{\bar{\theta}}^{\mathbf{nil}}"'] & \mathcal{W}_\Lambda(\bar{\theta}) \arrow[r] \arrow[d, "\varphi_{\bar{\theta}}"'] & \mathrm{diag}(\mathcal{W}_\Lambda(\bar{\theta})) \arrow[r] \arrow[d, "\varphi_{\bar{\theta}}^{\mathbf{diag}}"'] & 0 \\
0 \arrow[r] & \mathrm{Nil}(\mathcal{B}_\Lambda(\bar{\theta})) \arrow[r] & \mathcal{B}_\Lambda(\bar{\theta}) \arrow[r] & \mathrm{diag}(\mathcal{B}_\Lambda(\bar{\theta})) \arrow[r] & 0
\end{tikzcd}$$

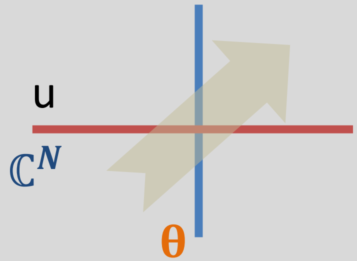
WBE: $\boxed{YW_\alpha[c] = \chi_\alpha}$

$$\mathbb{Q}_{a,s}(u) = \mathbb{Q}^0{}_{a,s}(u) + \boldsymbol{c_{a+1,s+1}}\mathbb{Q}_{a+1,s+1}(u)$$

What it has to do with spin chains?

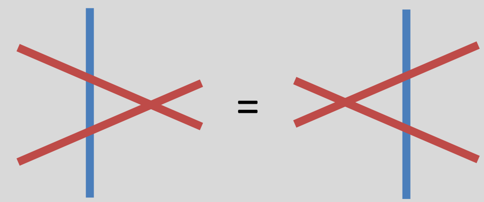
# Spin chain is a representation of Yangian

*Lax operator:* $L(u) \;=$

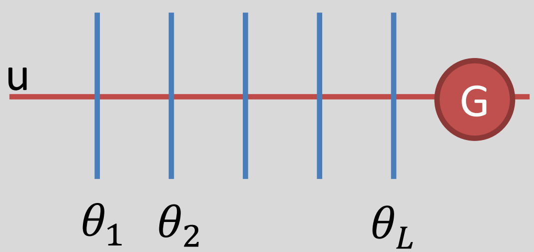



$$= (\boldsymbol{u} - \boldsymbol{\theta})\mathbf{1} + \hbar \sum_{i,j=1}^{m+n} (-\mathbf{1})^{|j|}\, \boldsymbol{E_{ij}} \otimes \boldsymbol{E_{ji}}$$

*Yangian:* $L(u)L(v)R(u-v) \;=\; R(u-v)\,L(v)\,L(u)$

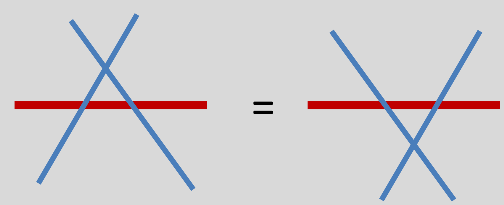



*Monodromy matrix:* $T(u) \;=$



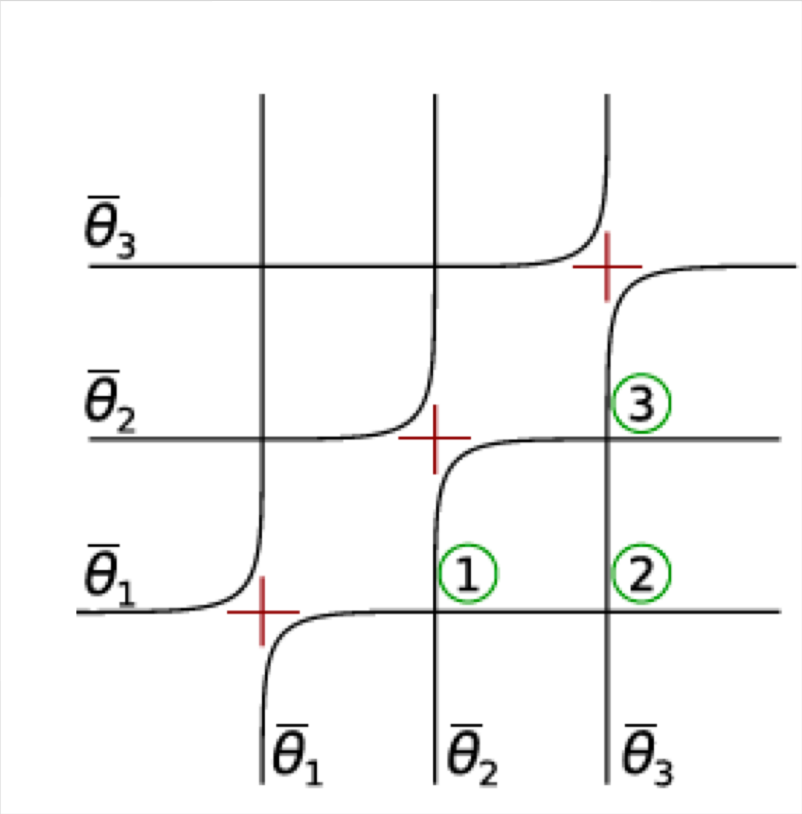

$u$ – *spectral parameter/rapidity*
$\boldsymbol{\theta_\alpha}$ - inhomogeneity ($\alpha = 1,2,\ldots,L$)
G – twist
$z_i$ - twist eigenvalues (i $= 1,2,\ldots,m+n$)

- It is irreducible iff $\theta_\alpha - \theta_\beta \neq \hbar$




- It is cyclic with HWS being cyclic vector iff $\theta_\alpha - \theta_\beta \neq \hbar$ for $\alpha > \beta$

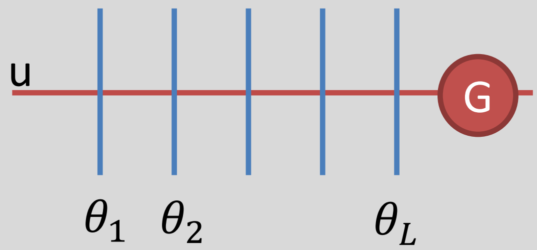
for gl(m|n): **[H.Zhang '14]** 1407.7001 [math.QA]



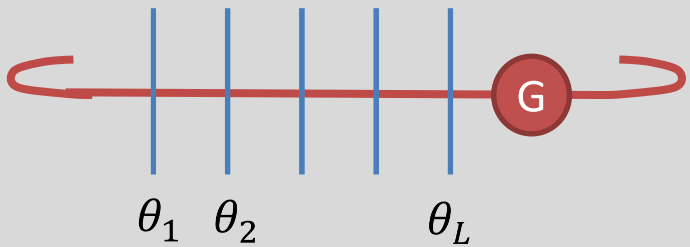
$\bar{\theta}_3$
$\bar{\theta}_2$
$\bar{\theta}_1$
3
1
2
$\bar{\theta}_1$
$\bar{\theta}_2$
$\bar{\theta}_3$

# Bethe algebra

*Monodromy matrix:* T(u) =




*Transfer matrix:* $\mathbb{T}(u) = \mathrm{Tr}\, T(u) =$




→ $[\mathbb{T}(u), \mathbb{T}(v)]=0$, *Bethe Ansatz to diagonalise …*

$$\mathcal{D} \equiv e^{-\frac{1}{2}\hbar\partial_u}$$

$$\mathrm{Ber}\,[\mathbb{1} - \mathcal{D}\,T(u)G\,\mathcal{D}] = \sum_{a=0}^{\infty}(-1)^a \mathcal{D}^a\,\mathbb{T}_{(1^a)}(u)\,\mathcal{D}^a$$

[Krichever, Lipan, Wiegmann, Zabrodin ’96]

[Nazarov ’91]

[Molev,Ragoucy ’09]

$$\begin{aligned}\mathbb{T}_{(s^a)} &= u^{a\,s\,L}\chi_{a,s}(G)(\mathbb{1} + \hat{d}_1\frac{\hbar}{u} + \ldots)\\ &\propto \frac{1}{Q^{[a-s]}_{\bar{\emptyset}|\bar{\emptyset}}}\prod_{k=1}^{a}\prod_{l=1}^{s}\frac{Q_{\bar{\emptyset}|\bar{\emptyset}}{}^{[a+s+2-2k-2l]}}{(\mathrm{Ber}\,G)^{u/\hbar}} \times \begin{cases}\epsilon^{b_1\ldots b_m} Q^{[\mathrm{m}-\mathrm{n}+s]}_{b_1\ldots b_a|\emptyset} Q^{[-s]}_{b_{a+1}\ldots b_m|\bar{\emptyset}}\,,\\ \epsilon^{i_1\ldots i_n} Q^{[\mathrm{m}-\mathrm{n}-a]}_{\emptyset|i_1\ldots i_s} Q^{[+a]}_{\bar{\emptyset}|i_{s+1}\ldots i_n}\,,\end{cases}\end{aligned}$$

[Gromov, Kazakov, Leurent, Tsuboi ’10]

[Tsuboi ’11]

# Bethe algebra

$$\mathbb{T}_{(s^a)} = u^{a\,s\,L}\chi_{a,s}(G)(\mathbb{1} + \hat{d}_1\frac{\hbar}{u} + \ldots)$$
$$\propto \frac{1}{Q^{[a-s]}_{\bar{\emptyset}|\bar{\emptyset}}} \prod_{k=1}^{a}\prod_{l=1}^{s} \frac{Q_{\bar{\emptyset}|\bar{\emptyset}}{}^{[a+s+2-2k-2l]}}{(\mathrm{Ber}\,G)^{u/\hbar}} \times \begin{cases} \epsilon^{b_1\ldots b_\mathsf{m}} Q^{[\mathsf{m}-\mathsf{n}+s]}_{b_1\ldots b_a|\emptyset} Q^{[-s]}_{b_{a+1}\ldots b_\mathsf{m}|\bar{\emptyset}}\,, \\ \epsilon^{i_1\ldots i_\mathsf{n}} Q^{[\mathsf{m}-\mathsf{n}-a]}_{\emptyset|i_1\ldots i_s} Q^{[+a]}_{\bar{\emptyset}|i_{s+1}\ldots i_\mathsf{n}}\,, \end{cases}$$

$$\longrightarrow$$

$$\mathbb{Q}_{a,s}(u) = u^{M_{a,s}} + \cdots$$

$$W\big(\mathbb{Q}_{a,s}, \mathbb{Q}_{a+1,s+1}\big) \propto \mathbb{Q}_{a+1,s}\mathbb{Q}_{a,s+1}$$

$$\mathbb{Q}_{a,s}(u) = \mathbb{Q}^0{}_{a,s}(u) + \boldsymbol{c}_{a+1,s+1}\mathbb{Q}_{a+1,s+1}(u)$$

$$\boxed{YW_\alpha[c] = \chi_\alpha}$$

$$\begin{array}{|c|c|c|c|} \hline \cancel{c_{1,1}} & \cancel{c_{1,2}} & \cancel{c_{1,3}} & c_{1,4} \\ \hline c_{2,1} & c_{2,2} & & \\ \hline c_{3,1} & & & \\ \hline \end{array}$$

- In summary: we know that there are commuting **operators** $\hat{c}$ that satisfy

$$\boxed{YW_\alpha[\hat{c}] = \chi_\alpha \times 1}$$

- We need to prove that the map $\varphi: c \mapsto \hat{c}$ is faithful and hence is an isomorphism.

# Part -1.b
# Introduction to polynomial rings, continued

$$\mathcal{W} = \frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{\langle P_1, P_2, \ldots, P_m \rangle}$$

**Thm:** # of solutions = dim $\mathcal{W}$

$\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_d$ - a basis of $\mathcal{W}$

$X$ - some element of $\mathcal{W}$

$$X\, \mathfrak{b}_a = \sum_b \check{X}_{ab}\, \mathfrak{b}_b$$

Regular representation

reg: $\mathcal{W} \mapsto \check{\mathcal{W}} \subset \mathrm{End}(\mathcal{W})$

reg: $X \mapsto \check{X}$

Example:

$$\mathcal{W} = \frac{\mathbb{C}[x]}{\langle x^2 - a\, x + b \rangle}$$

$(1, x)$ form a basis in $\mathcal{W}$

$x\, 1 = x$

$x\, x = \mathrm{a}\, x - b$

$$\check{x} = \begin{pmatrix} 0 & 1 \\ -b & a \end{pmatrix}$$

$$\mathcal{W} = \frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{\langle P_1, P_2, \dots, P_m \rangle}$$

Thm: # of solutions = dim $\mathcal{W}$

$\mathfrak{b}_1, \mathfrak{b}_1, \dots, \mathfrak{b}_d$ - a basis of $\mathcal{W}$

$X$ - Some element of $\mathcal{W}$

$$X\,\mathfrak{b}_a = \sum_b \check{X}_{ab}\,\mathfrak{b}_b$$

Regular representation

reg: $\mathcal{W} \to \check{\mathcal{W}} \subset \mathrm{End}(\mathcal{W})$

reg: $X \mapsto \check{X}$

- It is a faceful representation, $\mathcal{W}$ and $\check{\mathcal{W}}$ are isomorphic as algebras
- $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ is a solution iff $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ are eigenvalues of $\check{x}_1, \check{x}_2, \dots, \check{x}_n$

Proof:
- Each eigenvalue solves equations

$P(x_1, x_2, \dots, x_n)$=0 in $\mathcal{W}$ $\longleftrightarrow$ $P(\check{x}_1, \check{x}_2, \dots, \check{x}_n)$=0

(i.e. $P(x) = \sum_a q_a(x) P_a(x)$)

- Each solution is an eigenvalue

$$X\,\mathfrak{b}_a = \sum_b \check{X}_{ab}\,\mathfrak{b}_b \longrightarrow \bar{X}\,\bar{b}_a = \sum_b \check{X}_{ab}\,\bar{b}_b$$

$$\mathcal{W} = \frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{\langle P_1, P_2, \ldots, P_m \rangle}$$

Thm: # of solutions = dim $\mathcal{W}$

Degeneration case: dim $\mathcal{W}$ is number of solutions counted with multiplicity

Example:

$$\mathcal{W} = \frac{\mathbb{C}[x]}{\langle x^2 - a\,x + b\,\rangle}$$

$(1, x)$ form a basis in $\mathcal{W}$

$$x\,1 = x$$
$$x\,x = \mathrm{a}\,x - b$$

$$\check{x} = \begin{pmatrix} 0 & 1 \\ -b & a \end{pmatrix}$$

$a = b = 0$ $\qquad \check{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\qquad$ Jordan form, not diagonalisable!

General situation: $\check{\mathcal{W}}$ is a collection of commutative matrices

Dunford-Jordan-Chevalley decomposition: $\check{X} = D + N$

diagonal $\quad$ strictly upper triangular

Nil($\mathcal{W}$) – nilradical – subalgebra of all nilpotent elements (spanned by N)

diag($\mathcal{W}$):= $\mathcal{W}$/ Nil($\mathcal{W}$) (spanned by D)

$$\mathcal{W} = \frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{\langle P_1, P_2, \dots, P_m \rangle}$$

**Thm:** # of solutions = dim $\mathcal{W}$

Degeneration case: dim $\mathcal{W}$ is number of solutions counted with multiplicity

General situation: $\check{\mathcal{W}}$ is a collection of commutative matrices

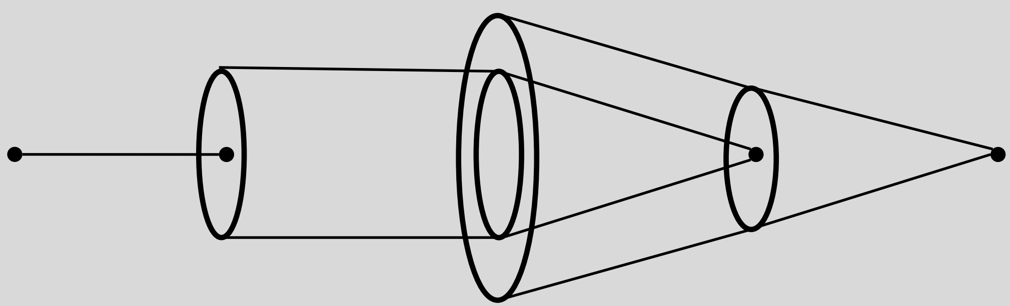
Dunford-Jordan-Chevalley decomposition: $\check{X} = D + N$

diagonal strictly upper triangular

Nil($\mathcal{W}$) – nilradical – subalgebra of all nilpotent elements (spanned by N)

diag($\mathcal{W}$):= $\mathcal{W}$/ Nil($\mathcal{W}$) (spanned by D)

$$0 \longrightarrow \mathrm{Nil}(\mathcal{W}) \longrightarrow \mathcal{W} \longrightarrow \mathrm{diag}(\mathcal{W}) \longrightarrow 0$$



- Bijection between distinct solutions and eigenspaces of $\check{\mathcal{W}}$
- $\check{\mathcal{W}}$ is a maximal commutative subalgebra of $\mathrm{End}(\mathcal{W})$

## Wronskian algebra $\mathcal{W}$

$$\mathcal{W} = \frac{\mathbb{C}[c]}{\langle YW_\alpha[c] - \bar{\chi}_\alpha \rangle}$$

## Bethe algebra $\mathcal{B}$

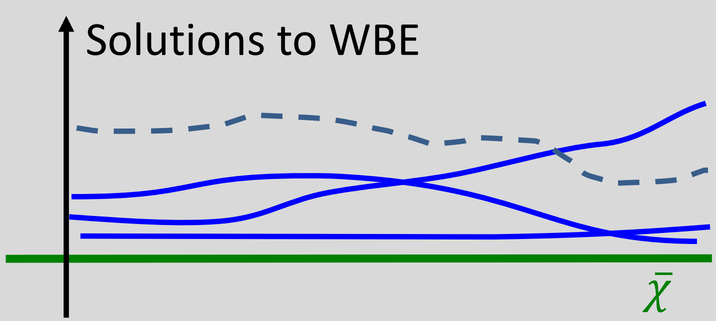
generated by operators $\hat{\boldsymbol{c}}$ that are known to satisfy WBE but could have other features

$$\varphi : c \mapsto \hat{\boldsymbol{c}}$$

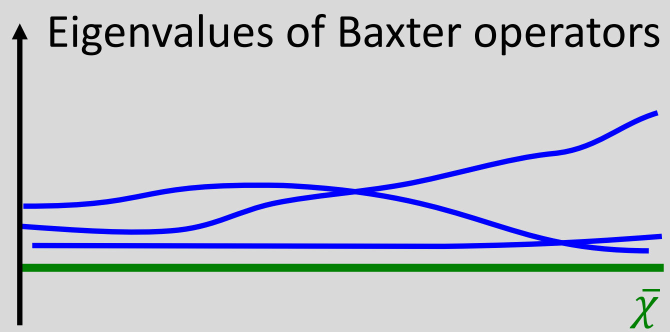
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Nil}(\mathcal{W}) \arrow[r] \arrow[d] & \mathcal{W} \arrow[r] \arrow[d] & \mathrm{diag}(\mathcal{W}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Nil}(\mathcal{B}) \arrow[r] & \mathcal{B} \arrow[r] & \mathrm{diag}(\mathcal{B}) \arrow[r] & 0
\end{tikzcd}$$

**Thm:** $\varphi_{diag}$ is always an isomorphism

[all solutions of WBE are eigenvalues of Baxter operators]




$\xrightarrow{\varphi}$




P(c) $\longrightarrow$ P($\hat{\boldsymbol{c}}$ )=0

## Wronskian algebra $\mathcal{W}$

$$\mathcal{W} = \frac{\mathbb{C}[c]}{\langle YW_\alpha[c] - \bar{\chi}_\alpha \rangle}$$

## Bethe algebra $\mathcal{B}$

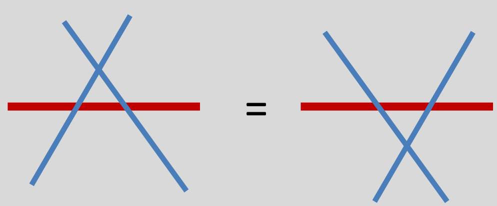
generated by operators $\hat{\boldsymbol{c}}$ that are known to satisfy WBE but could have other features

$$\varphi : c \mapsto \hat{\boldsymbol{c}}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Nil}(\mathcal{W}) \arrow[r] \arrow[d, "?"] & \mathcal{W} \arrow[r] \arrow[d] & \mathrm{diag}(\mathcal{W}) \arrow[r] \arrow[d, "\surd"] & 0 \\
0 \arrow[r] & \mathrm{Nil}(\mathcal{B}) \arrow[r] & \mathcal{B} \arrow[r] & \mathrm{diag}(\mathcal{B}) \arrow[r] & 0
\end{tikzcd}$$

**Thm:** $\varphi_{nil}$ is an isomorphism if $\theta_\alpha - \theta_\beta \neq \hbar$ for $\alpha > \beta$

(proof is based on adaptation of [Mukhin, Tarasov, Varchenko ’2013] + a bit more. The key ingredient is cyclicity of Yangian representation, and usage of dAHA – the Yangian centraliser)



$$S_\ell = \mathcal{P}_{\ell,\ell+1}\Pi_{\ell,\ell+1} - \frac{\hbar}{\theta_\ell - \theta_{\ell+1}}\left(\Pi_{\ell,\ell+1} - \mathbb{1}\right),$$

## Conclusions

- Bethe algebra is isomorphic to Wronskian algebra (algebra of polynomials) if $\theta_\alpha - \theta_\beta \neq \hbar$ for $\alpha > \beta$

  Remark: there is no restriction on values of $\bar{\chi}_\alpha$

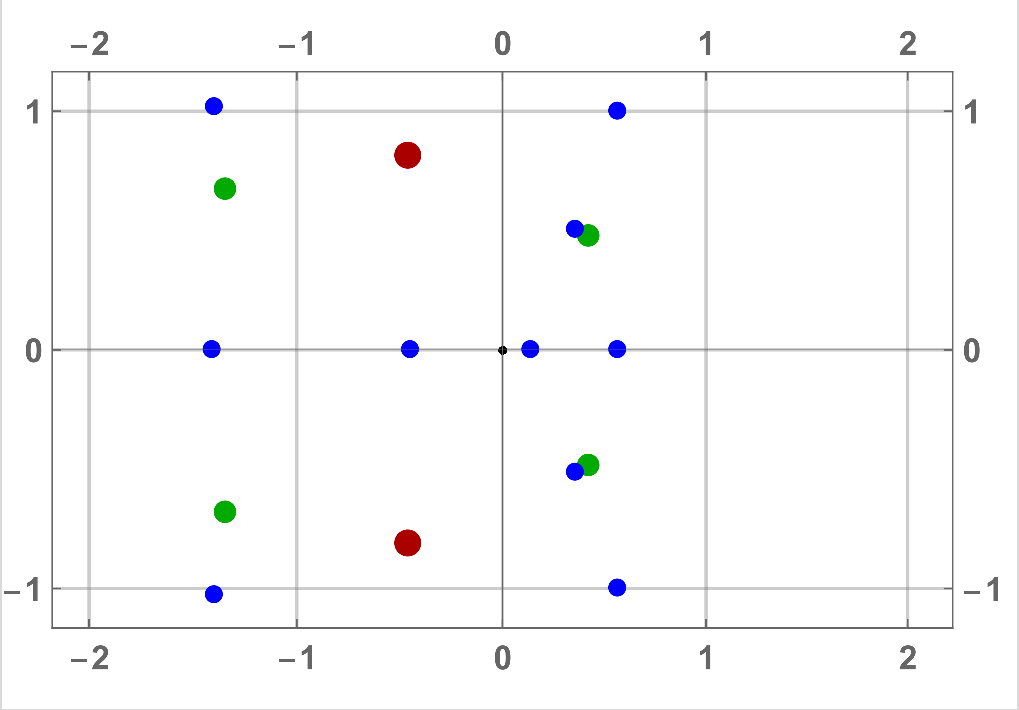
- There is a bijection between distinct solutions of WBE and eigenstates of the Bethe algebra

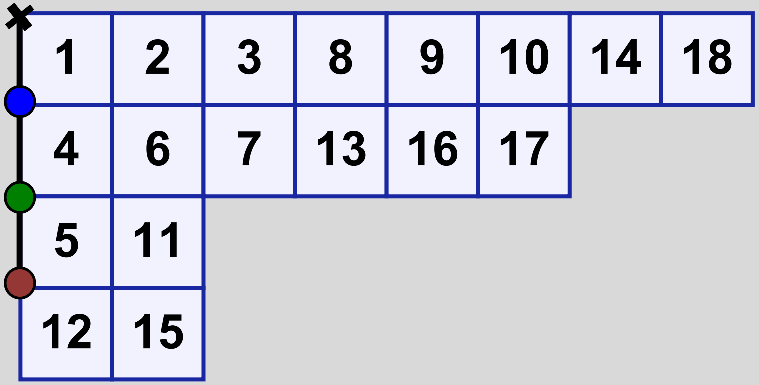
$$0 \longrightarrow \mathrm{Nil}(\mathcal{B}) \longrightarrow \mathcal{B} \longrightarrow \mathrm{diag}(\mathcal{B}) \longrightarrow 0$$

- Bethe algebra is maximal commutative

- Bethe algebra has simple spectrum if it is diagonalisable

# Part III
# Labelling with standard Young tableaux



- Bethe algebra has simple spectrum if it is diagonalisable

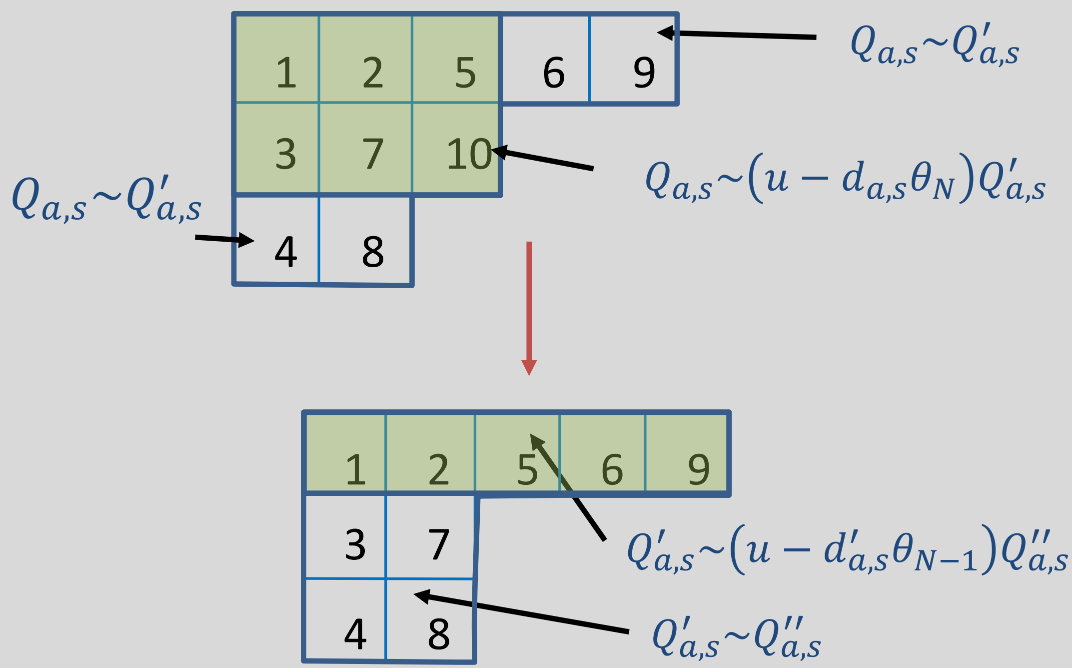
$\longrightarrow$

- Bethe algebra has simple spectrum if inhomogeneities are real!

New parameterisation of solutions:

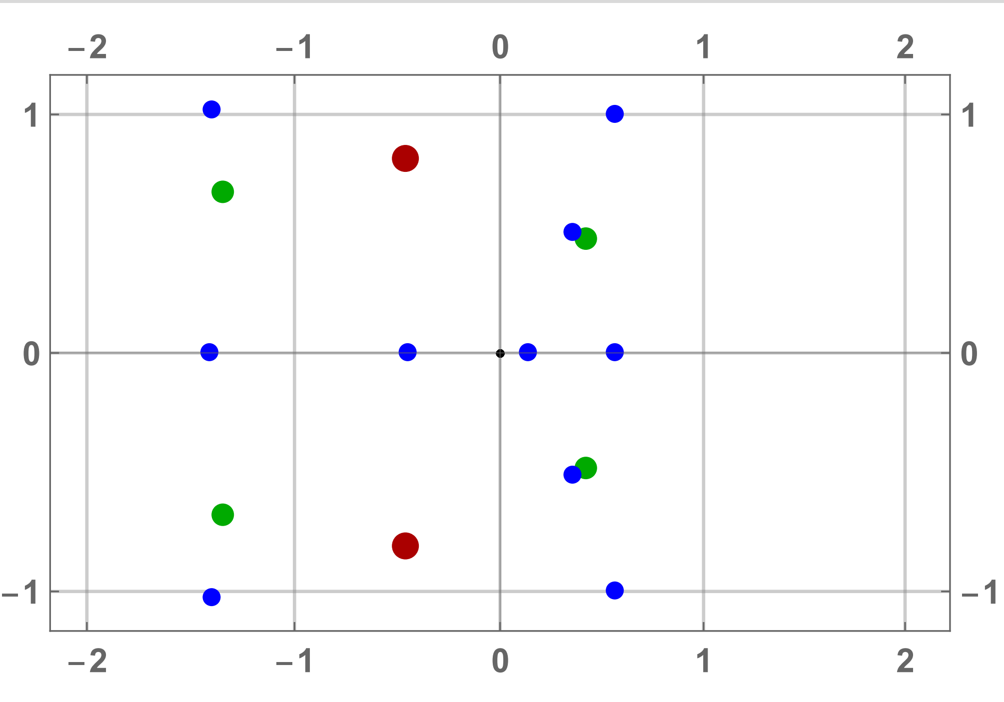
- the regime $\theta_L = \Lambda\,\theta_{L-1} = \Lambda^2\theta_{L-2} = \ldots, \Lambda \to \infty.$
- In this regime solutions are labelled (one-to-one) by Standard Young Tableaux.

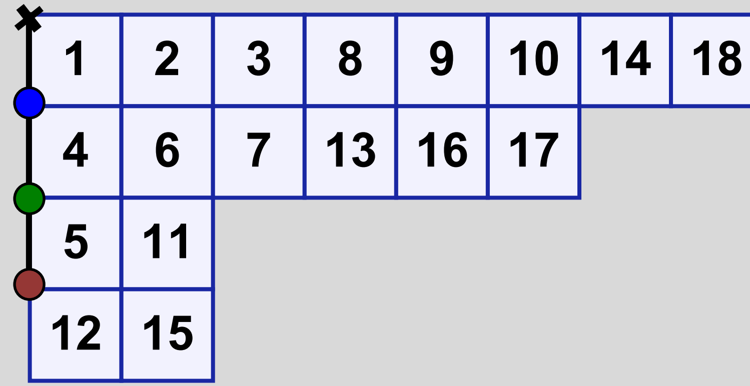
| | | | | |
|---|---|---|---|---|
| 1 | 2 | 5 | 6 | 9 |
| 3 | 7 | 10 | | |
| 4 | 8 | | | |



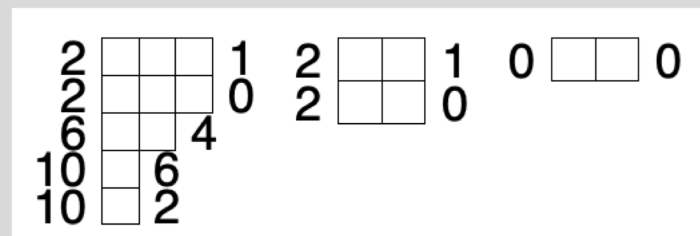

$$\mathbb{Q}_{a,s} \sim \prod_{\bar a>a,\bar s>s}\left(u-\bar\theta_{\mathcal{T}_{\bar a,\bar s}}\prod_{a'=1}^{a}\frac{h_{a',s}^{(\mathcal{T}_{\bar a,\bar s})}-1}{h_{a',s}^{(\mathcal{T}_{\bar a,\bar s})}}\prod_{s'=1}^{s}\frac{h_{a,s'}^{(\mathcal{T}_{\bar a,\bar s})}-1}{h_{a,s'}^{(\mathcal{T}_{\bar a,\bar s})}}\right)$$

…




Kerov-Kirillov-Reshetikhin bijection




Conjecture [exact meaning of string hyptohesis]: *Solutions of WBE labelled by SYT via continuation from the $\Lambda \to \infty$ limit coincide with solutions approximated by rigged configurations obtained via KKR bijection (when string hypothesis provides good approximation)*

Experimental evidence is planned to be published [Leurent, D.V. ’ 20xx.xxxx]

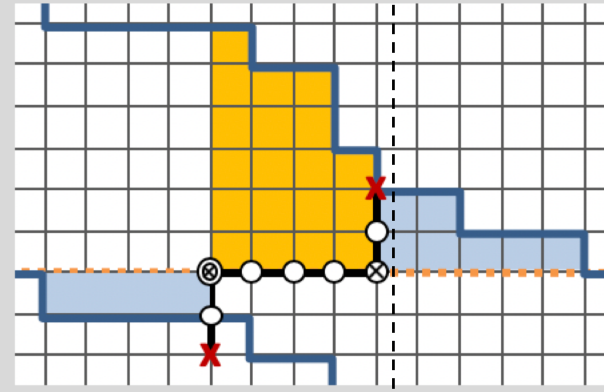
Thank you!

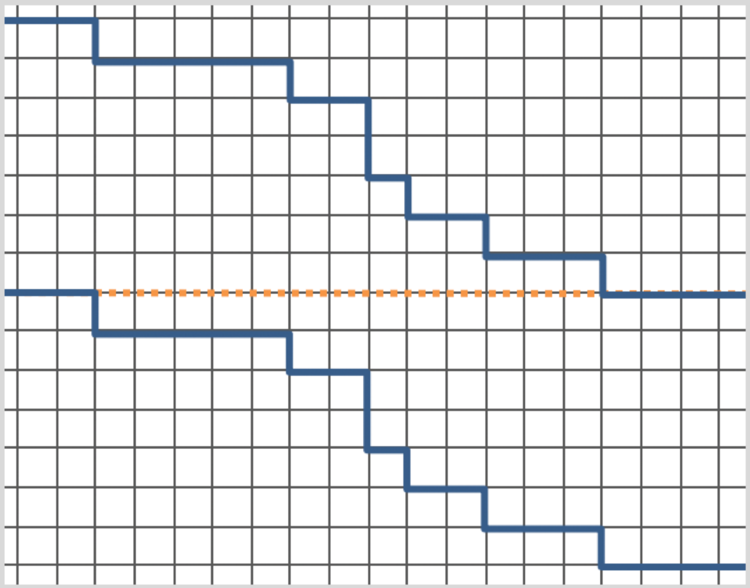
# Motivation N2: Representation theory

## $SU(p,q|m)$, extended and non-compact Young diagrams

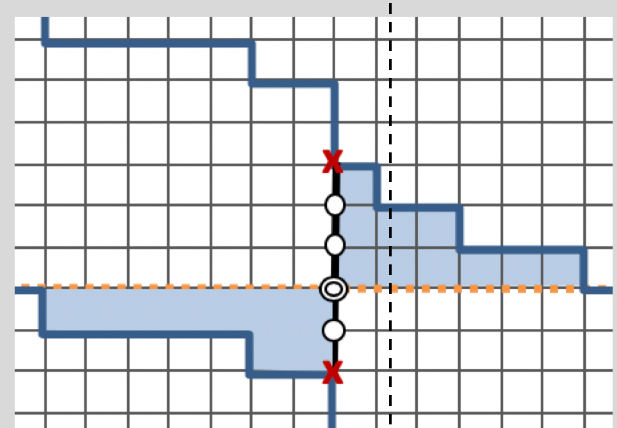
**[Günaydin, D.V. ’17]**
**[Marboe, D.V.’17]**

$SU(2,2|4)$

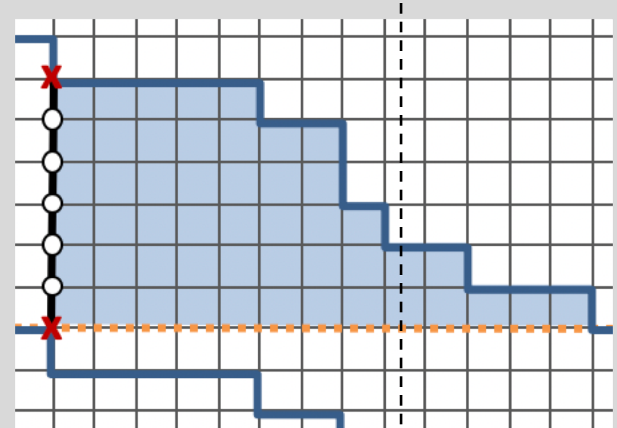




$SU(2,3)$

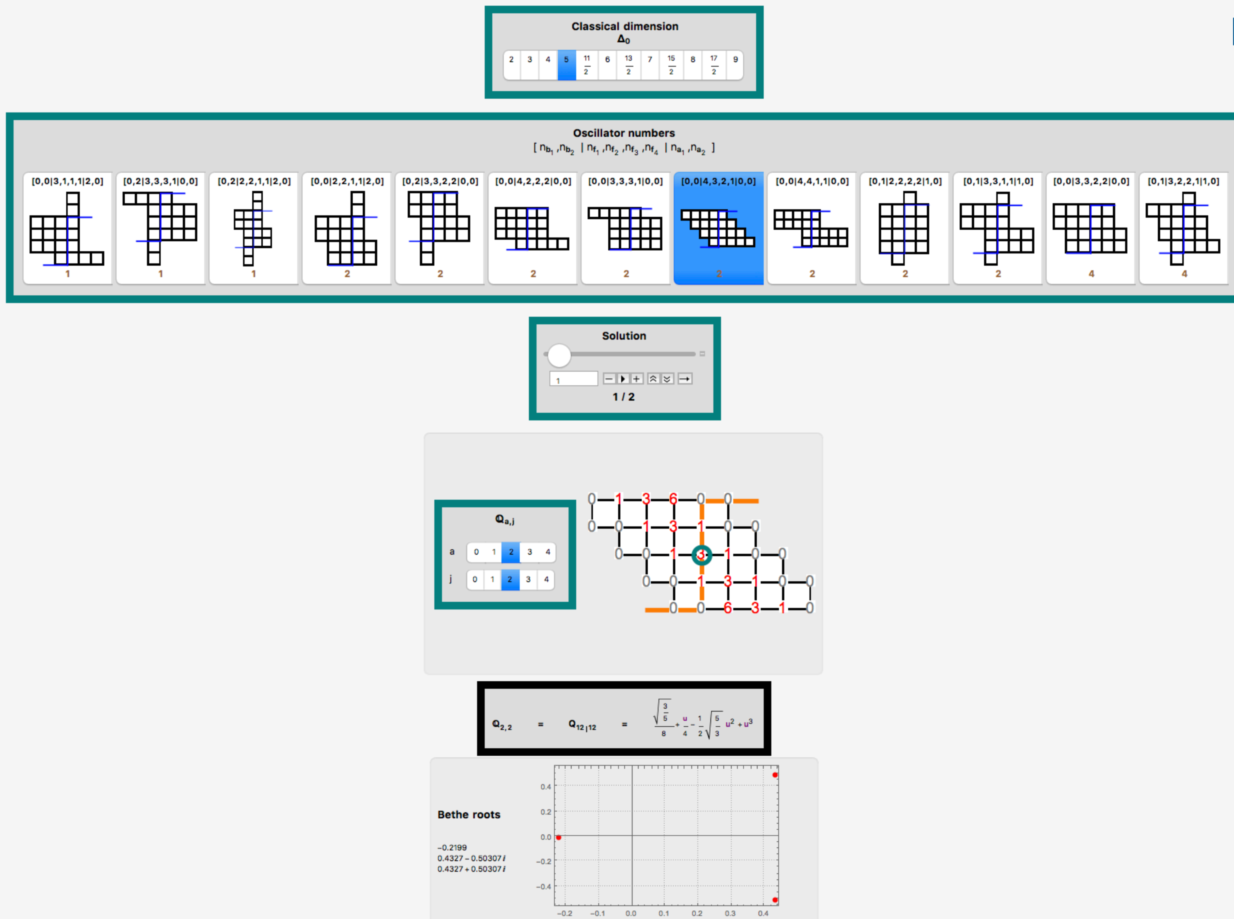


$SU(6)$



# ALL COMBINED… FULL PERTURBATIVE SPECTRUM OF ADS/CFT

[C.Marboe, D.V.'18]




**Anomalous dimension**
$\gamma$

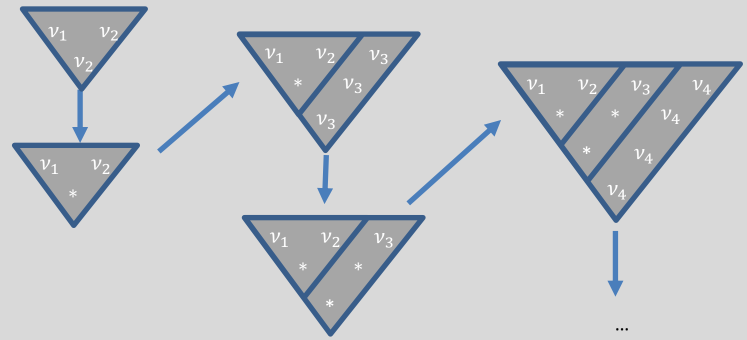
numerical

$$
\begin{aligned}
&10 g^2 - 30 g^4 + 200 g^6 + g^8(-1750 - 40\zeta_3) + g^{10}(17460 + 400\zeta_3 + 400\zeta_5) + g^{12}(-187560 - 4520\zeta_3 - 4280\zeta_5 - 4200\zeta_7) + g^{14}(2112020 + 54400\zeta_3 - 800\zeta_3^2 + 50720\zeta_5 + 71400\zeta_7 + 70560\zeta_9 - 46200\zeta_{11}) +\\
&g^{16}(-24583750 - 741120\zeta_3 + 6960\zeta_3^2 - 615000\zeta_5 + 170000\zeta_3\zeta_5 - 8000\zeta_5^2 - 835600\zeta_7 + 92400\zeta_3\zeta_7 - 28000\zeta_5\zeta_7 - 1357440\zeta_9 - 210000\zeta_3\zeta_9 - 1518000\zeta_{11} + 2059200\zeta_{13}) + g^{18}\Big(293375240 + 10600920\zeta_3 - 7360\zeta_3^2 + 240000\zeta_3^3 + 8029080\zeta_5 - \tfrac{408000}{7}\zeta_2^3\zeta_5 - 1960880\zeta_3\zeta_5 + 205000\zeta_3^2\zeta_5 - 1684000\zeta_5^2 -\\
&\quad 150000\zeta_3\zeta_5^2 + 9721200\zeta_7 + 122400\zeta_2^2\zeta_7 + \tfrac{192000}{7}\zeta_2^3\zeta_7 - 4254600\zeta_3\zeta_7 - 490000\zeta_3^2\zeta_7 - 802000\zeta_5\zeta_7 + 651000\zeta_7^2 + \tfrac{46558670\zeta_9}{3} + 4590000\zeta_2\zeta_9 - 12000\zeta_2^2\zeta_9 - 2800800\zeta_3\zeta_9 + 3060000\zeta_5\zeta_9 + \tfrac{52634750\zeta_{11}}{3} - 1650000\zeta_2\zeta_{11} + 7920000\zeta_3\zeta_{11} + 34895490\zeta_{13} - 57915000\zeta_{15} - 18000\zeta_5\zeta_{3,5} + 102000\zeta_{3,3,5} - 30000\zeta_{3,3,7} + 32400\zeta_{3,5,5}\Big) +\\
&g^{20}\Big(-3570237180 - 151641800\zeta_3 + 2299400\zeta_3^2 - \tfrac{8494400\zeta_3^3}{3} + 100000\zeta_3^4 - 113461000\zeta_5 + \tfrac{5817600}{7}\zeta_2^3\zeta_5 - \tfrac{1062400}{11}\zeta_2^5\zeta_5 + 17417400\zeta_3\zeta_5 + \tfrac{720000}{7}\zeta_2^3\zeta_3\zeta_5 - 12728000\zeta_3^2\zeta_5 - 850000\zeta_3^3\zeta_5 + 22834800\zeta_5^2 - 4230000\zeta_3\zeta_5^2 + \tfrac{5098000\zeta_5^3}{3} - 122563225\zeta_7 - 1745280\zeta_2^2\zeta_7 + \tfrac{20371200}{49}\zeta_2^3\zeta_7 - \tfrac{4079360}{7}\zeta_2^4\zeta_7 + 47013350\zeta_3\zeta_7 -\\
&\quad 216000\zeta_2^2\zeta_3\zeta_7 - 3369000\zeta_3^2\zeta_7 + 60926250\zeta_5\zeta_7 + 17200000\zeta_3\zeta_5\zeta_7 + 10344600\zeta_7^2 - \tfrac{1571524850\zeta_9}{9} - 65448000\zeta_2\zeta_9 - \tfrac{11985600}{7}\zeta_2^2\zeta_9 - \tfrac{13155200}{63}\zeta_2^3\zeta_9 + \tfrac{723077900\zeta_3\zeta_9}{9} - 8100000\zeta_2\zeta_3\zeta_9 + 14880000\zeta_3^2\zeta_9 + \tfrac{257956000\zeta_5\zeta_9}{9} - 55080000\zeta_7\zeta_9 - \tfrac{338094675\zeta_{11}}{2} - 46051500\zeta_2\zeta_{11} +\\
&\quad \tfrac{84700000}{3}\zeta_2^2\zeta_{11} + \tfrac{211674800\zeta_3\zeta_{11}}{3} - 100980000\zeta_5\zeta_{11} - \tfrac{7711516600\zeta_{13}}{21} + 430287000\zeta_2\zeta_{13} - 192192000\zeta_3\zeta_{13} - \tfrac{12765372200\zeta_{15}}{9} + 1322464000\zeta_{17} - 1038000\zeta_5\zeta_{3,5} + 444000\zeta_7\zeta_{3,5} + \tfrac{68000\zeta_5\zeta_{3,7}}{3} - 1454400\zeta_{3,3,5} - 180000\zeta_3\zeta_{3,3,5} - \tfrac{3183000\zeta_{3,3,7}}{7} + \tfrac{3520000\zeta_{3,3,9}}{9} + \tfrac{2366400\zeta_{3,5,5}}{7} + \tfrac{1162000\zeta_{5,3,7}}{3}\Big)
\end{aligned}
$$

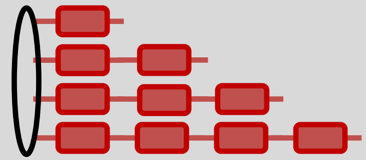
[Ryan, D.V. '18]
[Ryan, D.V. '20-to appear]

- A good SoV basis is constructed using the following formula:

$$\langle \mathbf{x}| = \langle \mathbf{\Omega}| \prod \det \hat{\boldsymbol{Q}}_{\circ}(\boldsymbol{x})$$



- It diagonalises quantum $\hat{B}(u)$

B = 

qdet M

$$= \prod (\boldsymbol{u} - \boldsymbol{X})$$

- It separates variables, and wave functions are naturally Baxter Q-functions – solutions of Baxter equation:

$$\mathbf{det}\left(\mathbf{1} + \boldsymbol{M}(\boldsymbol{u}) \boldsymbol{e}^{-\hbar \partial_u}\right) \boldsymbol{Q}_i = \mathbf{0}$$

- Recipe for construction of Bethe eigenstates:

$$|\mathbf{\Psi}\rangle = \prod \det Q_{\circ}(\boldsymbol{X}) |\mathbf{\Omega}'\rangle$$