Conformal Fourier Analysis and Gaudin Integrability

Based on work with Misha Isachenkov, Zhenya Sobko, Ilija Buric, Sylvain Lacroix, Jeremy Mann, Lorenzo Quintavalle ....

Conformal Partial Waves (CPWs)

Correlation functions of primary fields, weight Δ_i: (line, surface op.)

$$[P_\mu, \phi(x)] = \partial_\mu \phi(x)$$

$$[D, \phi(x)] = (x^\mu \partial_\mu + \Delta)\phi(x)$$

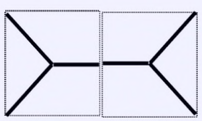
$\vdots$

$$\langle 0 | \prod_{i=1}^{4} \phi_i(x_i) | 0 \rangle = \Omega^{(N)}(x_{ij}, \Delta_i) \sum C_{12\phi^{(1)}} C_{3\phi^{(2)}\phi^{(3)}} \cdots C_{\phi^{(N-3)} N-1, N}\, G^{\Delta_i}_{\phi^{(i)}}(z)$$

OPE coefficients — $N-3$ intermediate fields — crossratios.

$N = 4$: 2
$N = 5$: + 3
$\vdots$ + $\vdots$
N: + d
$\leq Nd - \dim G$

Example: 4-pt function



$$= \quad G_{\Delta,l}(z,\bar{z}) \sim \qquad \mu = \frac{x_{10}}{x_{10}^2} - \frac{x_{20}}{x_{20}^2} \qquad \tilde{\mu} = \frac{x_{40}}{x_{40}^2} - \frac{x_{30}}{x_{30}^2}$$

$$\sim \int d^d x_0\, x_{10}^{l+a-\Delta} x_{20}^{l-a-\Delta} x_{30}^{l-b+\Delta-d} x_{40}^{l+b+\Delta-d} (|\mu||\tilde{\mu}|)^l\, Y_l^d\left(\frac{\mu\cdot\tilde{\mu}}{|\mu||\tilde{\mu}|}\right)$$

Zonal spherical functions

Can we compute CPWs a/as geodesic Witten diagrams?

↑ express in terms of known special functions

Casimir Equations & Integrability

$$\langle 0|\phi_1(x_1)\,\phi_2(x_2)\,\mathcal{C}^{(p)}_d\,P_{\Delta,l}\,\phi_3(x_3)\,\phi_4(x_4)|0\rangle = \frac{1}{2}C^{(p)}_{\Delta,l}\;\Omega^{DO}\,G_{\Delta,l}(z,\bar z)$$

↑ p-th order Casimir — ↑ eigenvalues in rep Δ, l.

$$= \mathrm{Cas}^{x\,(p)}_d \langle 0|\phi_1 \dots \phi_4|0\rangle$$

$$= \Omega\,\underbrace{\Omega^{-1}\,\mathrm{Cas}^{x\,(p)}_d\,\Omega}_{\mathrm{Cas}^{(p)}_d}\,G_{\Delta,l}(z,\bar z)$$

[Dolan, Osborn]

$$\mathrm{Cas}^2_d G(z,\bar z) = \frac{1}{2}C_{\Delta,l}G(z,\bar z)$$

$$C_{\Delta,l} = \Delta(\Delta-d) + l(l+d-2)$$

where

$$\mathrm{Cas}^2_d := D^2 + \overline{D}^2 + \epsilon\left[\frac{z\bar z}{\bar z - z}(\bar\partial - \partial) + (z^2\partial - \bar z^2\bar\partial)\right]$$

$$\epsilon = d-2 \qquad 2a = \Delta_2 - \Delta_1 \qquad 2b = \Delta_3 - \Delta_4$$

$$D^2 = z^2(1-z)\partial^2 - (a+b+1)z^2\partial - abz$$

Claim: [Isachenkov, VS]. Casimir equations for scalar 4-pt function

$\Leftrightarrow$ Schrödinger eq. for 2-particle trigonometric Calogero-Sutherland model.

$$\Psi_{\Delta,l}(u_1,u_2) = \omega^{IS}(z_i)\,G_{\Delta,l}(z_i) = \prod_{i=1,2}\frac{(z_i-1)^{\frac{a+b}{2}+\frac{1}{4}}}{z_i^{\frac{1}{2}+\frac{\epsilon}{2}}}\,|z_1-z_2|^{\frac{\epsilon}{2}}\,G_{\Delta,l}(z_1,z_2)$$

($z = z_1$, $\bar z = z_2$), $\quad z_i = \dfrac{1}{\sinh^2\frac{u_i}{2}}$

$$V(u_1,u_2) = \sum_{i=1}^{2}\left(\frac{(a+b)^2-1/4}{2\sinh^2 u_i} - \frac{ab}{2\sinh^2\frac{u_i}{2}}\right) + \frac{(d-2)(d-4)}{16\sinh^2\frac{u_1-u_2}{2}} + \frac{(d-2)(d-4)}{16\sinh^2\frac{u_1+u_2}{2}} + \frac{d^2-2d+2}{8}$$

Plan

Conformal Fourier Analysis & Integralility

Empirical evidence (N=4) ✓

Conceptual Explanation (N=4)

Discussion Break

Embedding into Gaudin Model

Lifting Correlators to Conformal Group

[Sobko; Buric, Isachenkov; VS]

$G = SO(1, d+1)$; special conformal trafo ; rotation

$$\Gamma_\phi \equiv \left\{ f : G \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} f(g\, e^{a^\mu K_\mu}) = f(g)\,;\; f(g\, R(\varphi)) = f(g) \\ f(g\, e^{\lambda D}) = e^{\lambda\Delta} f(g) \leftarrow \text{dilations.} \end{array} \right\}$$

Scalar primary field ϕ $\longmapsto$ line bundle Γ_ϕ over $G/P \cong S^d$

Spinning ... vector

$P = SO(1,1) \times SO(d) \ltimes \mathbb{R}^d$

Γ_ϕ carries action of conformal group G by left multiplication:

$$\text{e.g. } f(e^{\lambda D} e^{x^\mu P_\mu}) = f(e^{x^\mu \lambda P_\mu} e^{\lambda D}) = f(e^{\lambda x^\mu P_\mu}) e^{\lambda\Delta}$$

$[D, P_\mu] = P_\mu$; right covariance ; $\hat{=}\; x^\mu \partial_\mu + \Delta\,!$

$$\langle 0 | \prod \phi_i(x_i) | 0 \rangle \text{ "}\in\text{" } \left[\bigotimes_{i=1}^{N} \Gamma_{\phi_i}(G/P) \right]^G$$

function on $G \backslash (G/P)^{\otimes N}$

space of cross ratios!

4-point Function & Calogero Sutherland Model

Fact: [Dobrev et. al] $\Gamma_{\Delta_1}(G/P) \otimes \Gamma_{\Delta_2}(G/P) \simeq \Gamma_a(G/K)$

$2a = \Delta_2 - \Delta_1$ $\quad K = SO(1,1) \times SO(d)$

w. $\Gamma_a(G/K) = \{ f : G \to \mathbb{C} \mid f(gR) = f(g) ; f(ge^{\lambda D}) = e^{\lambda a} f(g) \}$

$\dim G/\!/K = 2\,!$

$$\Rightarrow \left[\bigotimes_{i=1}^{4} \Gamma_{\Delta_i}(G/P) \right]^G \simeq \left[\Gamma_a(G/K) \otimes \Gamma_b(G/K) \right]^G \simeq \Gamma_{a,b}(K \backslash G / K)$$

Radial Laplacians on double cosets $G/\!/K$ [Berezin, Helgason] are Calogero-Sutherland Hamiltonians [Olshanetski, Perelomov] [Feher, Pusztai] [Schomerus, Isachenkov, VS]

Remark: Product $\Omega^{DO} \omega^{LS}$ also has group theoretic interpretation!

relates $\langle \Pi \phi_i \rangle$ to CS wave function

→ Computable from Cartan dec.

Extensions: Spinning ϕ, boundaries, defects, supersymmetry...

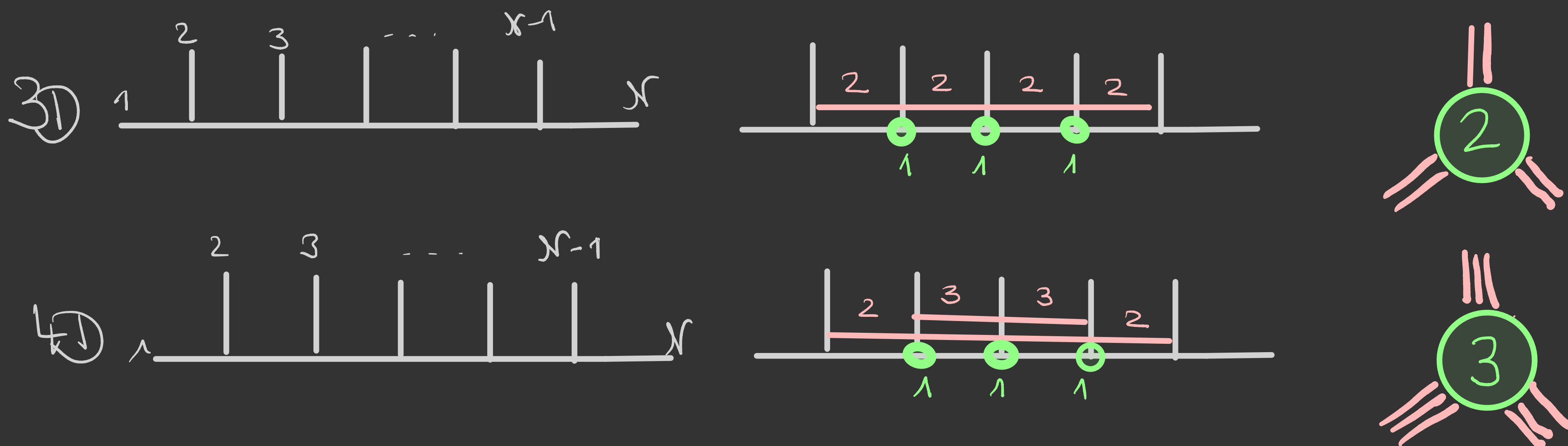
Generalization: N-Point Functions

Issue: $N = 5$, $d \geq 3$: $\langle 0 | \phi_1(x_1)\, \phi_2(x_2)\, P_{\Delta^{(1)}, \ell^{(1)}}\, \phi_3(x_3)\, P_{\Delta^{(2)}, \ell^{(2)}}\, \phi_4(x_4)\, \phi_5(x_5) | 0 \rangle$

→ 4 Dolan Osborn-like Casimir eqs. for function of 5 variables.
cross ratios.

Additional vertex differential operators required to characterize CPWs
measure tensor structures at vertex

more general: comb channel   degrees of freedom   maximal vertex



$(2N-5)!!$ different channels.

Gaudin Model and CPWs

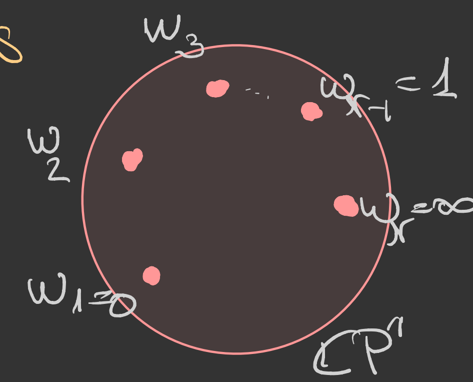
PRL 126, 021602

Gaudin model gives G-invariant integrable system on $T^*(G/P)^N$

Quantum Lax operators: differential operators acting on primary Φ_i

$$L_A(z) = \sum \frac{\mathcal{T}_A^{(i)}}{z - w_i}$$

$A = 1, \dots, \dim G$



Commuting families: $\mathcal{H}_p(z) = \kappa_p^{A_1 \dots A_p} L_{A_1}(z) \dots L_{A_p}(z) + \dots$

invariant symmetric tensor of order $p = 2, 4, 6 \dots$

$\rightsquigarrow \dim G \backslash (G/P)^N$ commuting Hamiltonians

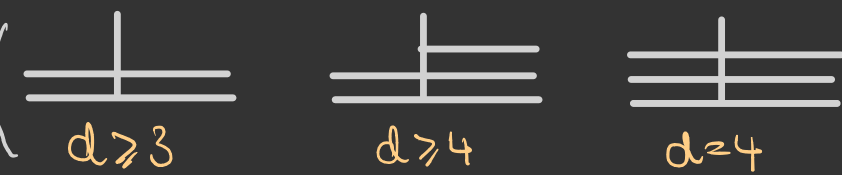
sufficiently many
depend on w_i!

Claim: For every OPE channel* there exists limit of parameters w_i in which Gaudin Hamiltonians include all Casimir operators + required number of novel vertex differential operators.

* to appear.

Example: Comb Channel Vertices in d=3,4

to appear

Claim: For vertices w. single d.o.f. (

)

vertex operator is Hamiltonian of elliptic Calogero-Sutherland integrable model discovered originally by Etingof et al.

very different context.

$$
\begin{aligned}
L = D^4 &+ [a_0\wp(z) + a_1\wp(z-\omega_1) - 2k(k+1)(\wp(z-\omega_2)+\wp(z-\omega_3))]D^2 \\
&+[b_0\wp'(z) + b_1\wp'(z-\omega_1) - 2k(k+1)(\wp'(z-\omega_2)+\wp'(z-\omega_3))]D \\
&+[k(k+1)(k+3)(k-2)(\wp^2(z-\omega_2)+\wp^2(z-\omega_3)) \\
&+k(k+1)(a_0-a_1)\wp(\omega_3)(\wp(z-\omega_2)-\wp(z-\omega_3)) \\
&+c_0\wp^2(z)+c_1\wp^2(z-\omega_1)]
\end{aligned}
$$

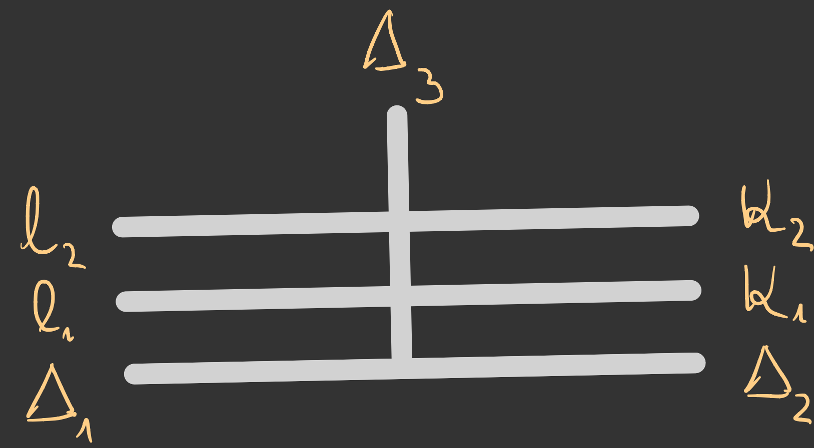
Weierstrass $\wp$ has half-periods

$\omega_1 = \frac{1}{2}(1+i)$ $\quad\omega_2 = \frac{i}{2}$ $\quad\omega_3 = \frac{1}{2}$

[Etingof, Felder, Ma, Veselov]

7 parameters.

d=4 Vertex



$$
\begin{aligned}
m_{10} &= \frac{7}{2} - l_1 - l_2 - 2\Delta_3 - 2(\kappa_1 - \kappa_2),\\
m_{20} &= \frac{7}{2} - l_1 - l_2 - 2\Delta_3 + 2(\kappa_1 - \kappa_2),\\
m_{30} &= -\frac{1}{2} + l_1 + l_2 + 2\Delta_3 + 2(\Delta_1 - \Delta_2),\\
m_{40} &= -\frac{1}{2} + l_1 + l_2 + 2\Delta_3 - 2(\Delta_1 - \Delta_2),\\
m_{11} &= -\frac{9}{2} - l_1 - l_2 + 2\Delta_3 - 2(\kappa_1 + \kappa_2),\\
m_{21} &= -\frac{9}{2} - l_1 - l_2 + 2\Delta_3 + 2(\kappa_1 + \kappa_2),\\
m_{31} &= -\frac{1}{2} + l_1 + l_2 - 2\Delta_3 + 2(\Delta_1 + \Delta_2),\\
m_{41} &= \frac{31}{2} + l_1 + l_2 - 2\Delta_3 - 2(\Delta_1 + \Delta_2).
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{4} m_{i,\nu} := 6,\\
&a_\nu := -11 + \sum_{1\le i<j\le 4} m_{i,\nu} m_{j,\nu},\\
&b_\nu := \frac{1}{2}\left(-a_\nu - 6 + \sum_{1\le i<j<k\le 4} m_{i,\nu} m_{j,\nu} m_{k,\nu}\right).\\
&c_\nu := \prod_{i=1}^{4} m_{i,\nu}.
\end{aligned}
$$

[Buric, Lacroix, Mann
Quintavalle, VS to appear]