QQ-system construction of $\mathfrak{so}(2r)$ spin chains

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Overview

Content

1. Review $sl(n)$ (A-type) spin chains
2. $so(2r)$ (D-type) spin chains
    - Basic introduction
    - Q-operator construction
    - QQ-system, Weyl-type formulas and QQ’-system
3. Systematic approach to Q-operators
4. Outlook

sl(n) spin chains

Rational sl(n) spin chains

Famous solution to YBE

$$R(x) = x + P \qquad \text{with} \qquad P = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}$$

$n^2 \times n^2$ matrix, x spectral parameter and $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$

Yang-Baxter equation

$$R_{12}(x-y)R_{13}(x-z)R_{23}(y-z) = R_{23}(y-z)R_{13}(x-z)R_{12}(x-y)$$

Rational sl(n) spin chains

Lax matrices

$$\mathcal{L}(x) = x + \sum_{i,j=1}^{r} e_{ij} \otimes J_{ji}$$

gl(n) commutation relations

$$[J_{ij}, J_{kl}] = \delta_{jk} J_{il} - \delta_{li} J_{kj}$$

equivalent to RLL

$$R(x-y)(\mathcal{L}(x) \otimes I)(I \otimes \mathcal{L}(y)) = (I \otimes \mathcal{L}(y))(\mathcal{L}(x) \otimes I)R(x-y)$$

Solution for any representation Λ

Evaluation map: Yangian $\mapsto$ Lie algebra

Transfer matrices

Commuting transfer matrices with diagonal twist

$$T_\Lambda(x) = \operatorname{tr}\left[\prod_{i=1}^{n} \tau_i^{J_{ii}}\right] \mathcal{L}_1(x)\mathcal{L}_2(x)\cdots\mathcal{L}_N(x)$$

with $\tau_i \in \mathbb{C}$ and T_Λ is $n^N \times n^N$ matrix

Focus on rectangular representations $\Lambda = (\underbrace{s,\ldots,s}_{a},0,\ldots,0)$

Hirota relations ($T^{[2k]} = T(x+k)$)

$$T_{(a,s)}^{[+1]} T_{(a,s)}^{[-1]} = T_{(a,s+1)} T_{(a,s-1)} + T_{(a-1,s)} T_{(a+1,s)}$$

Character limit $x \mapsto \infty$ such that $\chi_\Lambda = \operatorname{tr} \prod_{i=1}^{n} \tau_i^{J_{ii}}$

Algebraic Bethe ansatz

Fundamental transfer matrix for $a = s = 1$

$$T(x) = \operatorname{tr}_0 D_0 R_{01}(x) R_{02}(x) \cdots R_{0N}(x)$$

contains Hamiltonian

Eigenvalues from nested Bethe ansatz

$$T(x) = \sum_{k=1}^{n} \tau_k \frac{Q_{(k-1)}^{[k-r+2]}}{Q_{(k-1)}^{[k-r]}} \frac{Q_{(k)}^{[k-r-1]}}{Q_{(k)}^{[k-r+1]}} ,$$

with Q-functions $Q_{(i)}(x) = \prod_{k=1}^{m_i}(x - x_k^{(i)})$

- $Q_{(0)} = I$ and $Q_{(n)} = x^N$
- Magnon numbers m_i
- Bethe roots $x_k^{(i)}$ satisfy Bethe equations
- Bethe equations arise as pole cancelling conditions

Q-operators

Q-operator are constructed from ∞-dim representation in auxiliary space [Bazhanov,Lukyanov,Zamolodchikov],[Bazhanov,RF,Lukowski,Meneghelli,Staudacher]

$$Q_{(k)}(x) = \operatorname{tr} D_{(k)} L_{(k)}(x) \otimes \ldots \otimes L_{(k)}(x)$$

with $k = 1, \ldots, n-1$ and

$$L_{(k)}(x) = \left(\begin{array}{c|c} x I_k - \bar{\mathbf{A}}\mathbf{A} & \bar{\mathbf{A}} \\ \hline -\mathbf{A} & I_{n-k} \end{array}\right),$$

Submatrices $\mathbf{A}_{\dot\alpha\alpha} = a_{\dot\alpha\alpha}$ and $\bar{\mathbf{A}}_{\alpha\dot\alpha} = \bar{a}_{\alpha\dot\alpha}$ with $[a_{\dot\alpha\alpha}, \bar{a}_{\beta\dot\beta}] = \delta_{\alpha\beta}\delta_{\dot\alpha\dot\beta}$

Indices: $\alpha, \beta = 1, \ldots, k$ and $\dot\alpha, \dot\beta = k+1, \ldots, n$

- Solutions to RLL

  $$R(x-y)(L_{(k)}(x)\otimes I)(I\otimes L_{(k)}(y)) = (I\otimes L_{(k)}(y))(L_{(k)}(x)\otimes I)R(x-y)$$

- Classification in terms of quiver diagrams [RF,Pestun]

More Q-operators

R-matrix is $gl(n)$ invariant

$$[R(x), B \otimes B] = 0$$

Generate further solutions by permuting x's and 1's

$$\operatorname{diag} L_{(k)}(x) \sim (\underbrace{x, \ldots, x}_{k}, \underbrace{1, \ldots, 1}_{n-k})$$

For fixed k we find $\binom{n}{k}$ algebraically independet solutions

Label solutions by set $I \subseteq \{1, \ldots, n\}$ denoting positions of x's , e.g.

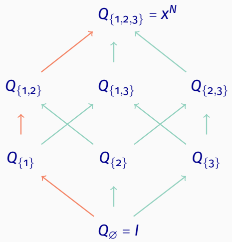
$$L_{\{1,2,\ldots,k\}}(x) = L_{(k)}(x)$$

This leads to $\binom{n}{k}$ Q-operators Q_I at level $|I| = k$

Nesting path and Hasse diagram

Q-functions in eigenvalue equation for T depend on the choice of nesting path

- In total there are $n!$ nesting paths
- Conveniently depicted in Hasse diagram



Hasse diagram for A_2

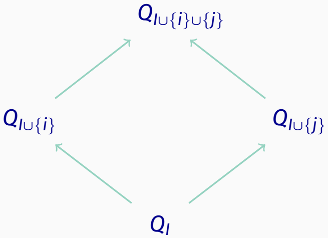
Change of path described by QQ-relations

QQ-relations and determinant formula

QQ-relations at each plaquette

$$Q_I Q_{I\cup\{i\}\cup\{j\}} = Q^{[+]}_{I\cup\{i\}} Q^{[-]}_{I\cup\{j\}} - Q^{[-]}_{I\cup\{i\}} Q^{[+]}_{I\cup\{j\}}$$

with $i \neq j$ and $i, j \notin I$



Allow to express transfer matrices in terms of single index Q's

Determinant formula for arbitrary Λ

$$T_\Lambda(x) = \det_{1\le i<j\le n} Q_{\{i\}}(x + \lambda_j + n - j)$$

so(2r) spin chains

Fundamental R-matrix

Fundamental R-matrix for $so(2r)$

$$R(x) = x(x+\kappa)I + (x+\kappa)P - xQ$$

with $\kappa = r-1$

Here $P = \sum_{i,j=1}^{2r} e_{ij} \otimes e_{ji}$ and $Q = \sum_{i,j=1}^{2r} e_{ij} \otimes e_{i'j'}$ where $i' = 2r - i + 1$

Remark: Similarity transformation leads Zamolodchikov form

$$\tilde{R}(x) = (S \otimes S)R(x)(S^{-1} \otimes S^{-1}) = x(x+\kappa)I + (x+\kappa)P - xK$$

with $K = \sum_{i,j=1}^{2r} e_{ij} \otimes e_{ij}$ and

$$S = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} -\mathrm{i}J & J \\ \hline \mathrm{i}I & I \end{array} \right), \qquad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathrm{i}^2 = -1$$

Fundamental transfer matrix

Transfer matrix

$$T(x) = \operatorname{tr} D_a \, R_{a1}(x) R_{a2}(x) \cdots R_{aN}(x)$$

is $(2r)^N \times (2r)^N$ matrix with diagonal twist

$$D = \operatorname{diag}(\tau_1, \ldots, \tau_r, \tau_r^{-1}, \ldots, \tau_1^{-1})$$

with $\tau_i = \tau_{i'}^{-1}$.

Logarithmic derivative at permutation point

$$H = \sum_{k=1}^{N} \mathcal{H}_{k,k+1} \qquad \text{with} \qquad \mathcal{H}_{k,k+1} = \kappa^{-1} \left(I - Q + \kappa P\right)_{k,k+1}$$

Twisted boundary conditions: $\mathcal{H}_{N,N+1} = D_N \mathcal{H}_{N,1} D_N^{-1}$

Bethe ansatz

Transfer matrix eigenvalue[Reshetikhin; de Vega, Karowski]

$$T(x) = Q_{(0)}^{[1-r]} Q_{(0)}^{[r-1]} \sum_{k=1}^{r} \left[\tau_k \frac{Q_{(k-1)}^{[k-r+2]}}{Q_{(k-1)}^{[k-r]}} \frac{Q_{(k)}^{[k-r-1]}}{Q_{(k)}^{[k-r+1]}} + \tau_k^{-1} \frac{Q_{(k-1)}^{[r-k-2]}}{Q_{(k-1)}^{[r-k]}} \frac{Q_{(k)}^{[r-k+1]}}{Q_{(k)}^{[r-k-1]}} \right] ,$$

with $Q_{(0)} = x^N$, $Q_{(r-1)} = S_{(+)} S_{(-)}$ and $Q_{(r)} = S_{(+)}^{[+1]} S_{(+)}^{[-1]}$.

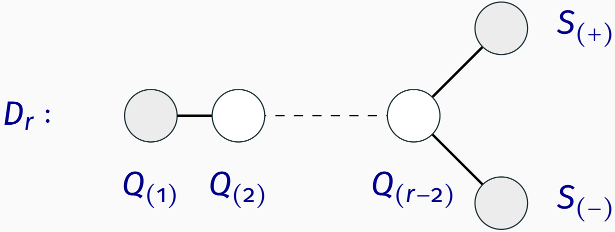
Q-functions

$$Q_{(k)} = \prod_{i=1}^{m_k} (x - x_i^{(k)}) , \qquad k = 1, \ldots, r-2$$

$$S_{(\pm)} = \prod_{i=1}^{m_\pm} (x - x_i^{(\pm)}) ,$$

Degree of polynomial Q-functions m_k determined on a given state of weight $\vec{n}$ via $\vec{n} = \Lambda - \vec{m} A$ where A is the Cartan matrix

Dynkin diagram



Q-functions can be associated to nodes of Dynkin diagram

Bethe equations

Bethe equations couple roots on neighboring nodes

$$\frac{\tau_k}{\tau_{k+1}} = \left(\frac{Q_{(k-1)}^{[-1]}}{Q_{(k-1)}^{[+1]}} \frac{Q_{(k)}^{[+2]}}{Q_{(k)}^{[-2]}} \frac{Q_{(k+1)}^{[-1]}}{Q_{(k+1)}^{[+1]}} \right)_k , \qquad (k = 1, 2, \ldots, r-3)$$

$$\frac{\tau_{r-2}}{\tau_{r-1}} = \left(\frac{Q_{(r-3)}^{[-1]}}{Q_{(r-3)}^{[+1]}} \frac{Q_{(r-2)}^{[+2]}}{Q_{(r-2)}^{[-2]}} \frac{S_{(+)}^{[-1]}}{S_{(+)}^{[+1]}} \frac{S_{(-)}^{[-1]}}{S_{(-)}^{[+1]}} \right)_{r-2} ,$$

$$\frac{\tau_{r-1}}{\tau_r} = \left(\frac{Q_{(r-2)}^{[-1]}}{Q_{(r-2)}^{[+1]}} \frac{S_{(+)}^{[+2]}}{S_{(+)}^{[-2]}} \right)_+ ,$$

$$\frac{\tau_{r-1}}{\tau_r} = \left(\frac{Q_{(r-2)}^{[-1]}}{Q_{(r-2)}^{[+1]}} \frac{S_{(-)}^{[+2]}}{S_{(-)}^{[-2]}} \right)_- ,$$

Plücker relations

Replace $Q_k \to \widetilde{Q}_k$ in TQ- and Bethe equations

Tail:

$$Q_{k-1}Q_{k+1} = \sqrt{\frac{\tau_{k+1}}{\tau_k}}Q_k^+\widetilde{Q}_k^- - \sqrt{\frac{\tau_k}{\tau_{k+1}}}Q_k^-\widetilde{Q}_k^+ , \qquad k = 1, \dots , r-3$$

Fork:

$$Q_{r-3}\, S_+\, S_- = \sqrt{\frac{\tau_{r-2}}{\tau_{r-1}}}Q_{r-2}^+\widetilde{Q}_{r-2}^- - \sqrt{\frac{\tau_{r-1}}{\tau_{r-2}}}Q_{r-2}^-\widetilde{Q}_{r-2}^+ .$$

Spinor:

$$Q_{r-2} = \sqrt{\frac{\tau_{r-1}}{\tau_r}}S_+^+\widetilde{S}_+^- - \sqrt{\frac{\tau_r}{\tau_{r-1}}}S_+^-\widetilde{S}_+^+ , \qquad Q_{r-2} = \sqrt{\frac{\tau_{r-1}}{\tau_r}}S_-^+\widetilde{S}_-^- - \sqrt{\frac{\tau_r}{\tau_{r-1}}}S_-^-\widetilde{S}_-^+ .$$

Spinorial relations appeared in [Masoero,Raimondo,Valeri]

Transfer matrices

Construction of transfer matrix $T_{(a,s)}$ for rectangular representations more involved

- Evaluation map does not exist
- Closed-form expression for Lax matrices unknown

Lax matrices known explicitly 1-st fundamental and spinorial representations [Reshetikhin; Shankar, Witten]

Can be written in terms of generators of $so(2r)$

$$[F_{ij}, F_{kl}] = \delta_{jk} F_{il} - \delta_{i'k} F_{j'l} - \delta_{jl'} F_{ik'} + \delta_{il} F_{j'k'} \,,$$

with $F_{ij} = -F_{j'i'}$, $i, j, k, l = 1, \ldots, 2r$ and $i' = 2r - i + 1$

Cartan elements F_{ii} with $1 \leq i \leq r$

Fundamental Lax matrix

Quadratic Lax matrix corresponding to 1-st fundamental

$$\mathcal{L}_{(1,s)}(x) = x^2 I + x \sum_{i,j=1}^{2r} F_{ij} \otimes E_{ji} + \sum_{i,j=1}^{2r} G_{ij} \otimes E_{ji} \, .$$

with

$$G_{ij} = \frac{1}{2} \sum_{k=1}^{2r} F_{kj} F_{ik} + \frac{\kappa}{2} F_{ij} - \frac{1}{4} \left((\kappa - 1)^2 + 2\kappa s + s^2 \right) \delta_{ij} \, .$$

Generators of $so(2r)$ in "one-row" representation

$$\Lambda = (s, 0, \ldots, 0)$$

RLL relies on characteristic identity

$$\sum_{j,k=1}^{2r} \left(F_{ij} - \delta_{ij} \right) \left(F_{jk} + s\delta_{jk} \right) \left(F_{kl} - (s + 2\kappa) \delta_{kl} \right) = 0$$

Recover R-matrix for $s = 1$: $F_{ij} = e_{ij} - e_{j'i'}$

Spinorial Lax matrix

Linear Lax matrix corresponding to spinor nodes ±

$$\mathcal{L}_{(\pm,s)}(z) = z + \sum_{i,j=1}^{2r} E_{ij} \otimes F_{ji}$$

with F_{ij} in spinorial representation

$$\Lambda = (s/2, \ldots, s/2, \pm s/2)$$

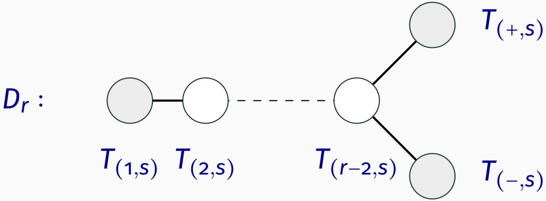
Characteristic identity

$$\sum_{j=1}^{2r} \left(F_{ij} + s\delta_{ij}\right)\left(F_{jk} - (s+\kappa)\delta_{jk}\right) = 0\,.$$

Can be realised in terms of gamma matrices [Shankar,Witten]

Further interesting studies of RLL [Fuksa,Karakhanyan,Kirschner,Isaev]

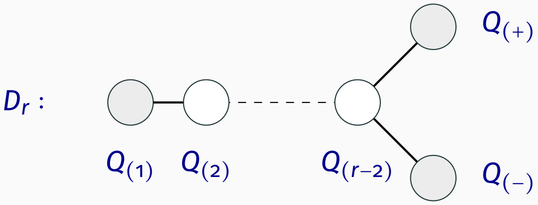
Transfer matrices



- Can construct transfer matrices at the grey nodes
- For the white nodes the representation of Lie algebra does not lift to Yangian
- Yangian described by Kirillov-Reshetikhin modules
- Example: $\chi_{KR}(2,s) = \chi(2,s) + \chi(2,s-1) + \ldots + \chi(2,0)$
- Can obtain $T_{a,s}$ for $a \neq 1, \pm$ from Hirota equations

[Kuniba,Suzuki,Nakanishi]

Q-operators



Similarly for Q-operators

- Oscillator construction only known for extremal nodes [RF]

Q-operators from oscillator realisation

Lax matrices for Q-operators at the first node

$$L_{(1)}(z) = \left(\begin{array}{c|c|c} z^2 + z(2 - r - \bar{w}w) + \frac{1}{4}\bar{w}J\bar{w}^t w^t J w & z\bar{w} - \frac{1}{2}\bar{w}J\bar{w}^t w^t J & -\frac{1}{2}\bar{w}J\bar{w}^t \\ \hline -zw + \frac{1}{2}J\bar{w}^t w^t J w & zI - J\bar{w}^t w^t J & -J\bar{w}^t \\ \hline -\frac{1}{2}w^t J w & w^t J & 1 \end{array}\right).$$

with $2r - 2$ pairs of oscillators $[a_i, \bar{a}_j] = \delta_{ij}$ and

$$\bar{w} = (\bar{a}_2, \ldots, \bar{a}_r, \bar{a}_{r'}, \ldots, \bar{a}_{2'}), \qquad w = (a_2, \ldots, a_r, a_{r'}, \ldots, a_{2'})^t.$$

Definition of Q-operator

$$Q_{(1)}(x) = \operatorname{tr} D_{(1)} L_{(1)}(x) \otimes \ldots \otimes L_{(1)}(x)$$

Q-operators from oscillator realisation

Lax matrix for Q-operators at the spinorial node

$$L_+(z) = \left(\begin{array}{c|c} z\,I - \bar{\mathbf{A}}\mathbf{A} & \bar{\mathbf{A}} \\ \hline -\mathbf{A} & I \end{array}\right),$$

with

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{a}_{-r,1} & \cdots & \bar{a}_{-r,r-1} & 0 \\ \vdots & \iddots & 0 & -\bar{a}_{-r,r-1} \\ \bar{a}_{-2,1} & 0 & \iddots & \vdots \\ 0 & -\bar{a}_{-2,1} & \cdots & -\bar{a}_{-r,1} \end{pmatrix}, \qquad \mathbf{A} = \bar{\mathbf{A}}^\dagger$$

Definition of Q-operator

$$S_{(+)}(x) = \operatorname{tr} D_+ L_+(x) \otimes \ldots \otimes L_+(x)$$

Q-operators

Q-operators commute with transfer matrix by construction

$$\left[Q_{(1)}(x), T(y)\right] = 0 = \left[S_{(+)}(x), T(y)\right]$$

Can be evaluated explicitly, e.g. $Q_{(1)}$ for $N = 1$:

$$\begin{aligned}(Q_1(x))_{11} = x^2 - x \sum_{k=2}^{2r-1} \left(\frac{1}{2} + \frac{\tau_k}{\tau_1 - \tau_k}\right) \\ + \frac{1}{2} \sum_{k=2}^{2r-1} \left[\frac{1}{(\tau_1 - \tau_k)(\tau_1 - \tau_k^{-1})} + \frac{\tau_k}{2(\tau_1 - \tau_k)}\right] + \frac{2r-3}{4}\,,\end{aligned}$$

$$(Q_1(x))_{ii} = x - \frac{1}{2} + \frac{\tau_1^{-1}}{\tau_1^{-1} - \tau_i}\,, \qquad 1 < i \leq r\,,$$

$$(Q_1(x))_{ii} = x + \frac{1}{2} - \frac{\tau_{i'}}{\tau_1 - \tau_{i'}}\,, \qquad r < i \leq 2r-1\,,$$

$$(Q_1(x))_{2r2r} = 1\,.$$

More Q-operators

More Q-operators by permuting spectral parameters on the diagonal of Lax matrix

- $so(2r)$-invariance $[R(x), B \otimes B] = 0$ for $BB' = I$
- $2r$ Q-operators from $\operatorname{diag} L_{(1)} \sim (z^2, z, \ldots, z, 1)$:

$$Q_{\{i\}} \quad \text{with} \quad i \in \{1, \ldots, 2r\}$$

- $2 \times 2^{r-1}$ Q-operators from $\operatorname{diag} L_{(+)} \sim (z, \ldots, z, 1 \ldots, 1)$:

$$S_{\vec{\alpha}=(\alpha_1,\ldots,\alpha_r)} \quad \text{with} \quad \alpha_i = \pm 1 \text{ with } \prod \alpha_i = \pm 1 \text{ for } \vec{\alpha}_\pm$$

Alternative labelling:

$S_{\{i_1,\ldots,i_r\}}$ such that $\alpha_{i_k} = 1$ if $i_k \leq r$ and else $\alpha_{i_k} = -1$

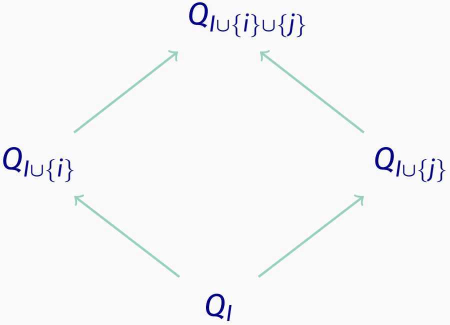
Example: $S_{(+,-,+,+)} = S_{\{1,2',3,4\}}$

QQ-relations along the tail

QQ-relations along the tail as for A-type

$$Q^{[+1]}_{J\cup\{i\}} Q^{[-1]}_{J\cup\{j\}} - Q^{[-1]}_{J\cup\{i\}} Q^{[+1]}_{J\cup\{j\}} = \frac{\tau_i - \tau_j}{\sqrt{\tau_i \tau_j}} Q_J Q_{J\cup\{i,j\}}$$

but exclude Q-functions with $i, i' \in I$

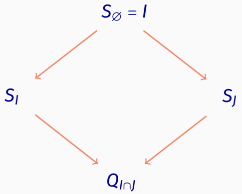


Spinorial QQ-relations

For S_I with $I = \{i_1, \dots, i_r\}$ and S_J with $J = \{i_1, \dots, i_{r-2}, i'_{r-1}, i'_r\}$

$$S_I^{[+1]} S_J^{[-1]} - S_I^{[-1]} S_J^{[+1]} = \frac{\tau_{i_{r-1}} \tau_{i_r} - 1}{\sqrt{\tau_{i_{r-1}} \tau_{i_r}}} Q_{I \cap J}$$

where $I \cap J = \{i_1, \dots, i_{r-2}\}$



Example D_5:

$I = \{1, 5, 3', 2, 4\} \leftrightarrow \vec{\alpha} = (+1, +1, -1, +1, +1)$

$J = \{1, 5, 3', 2', 4'\} \leftrightarrow \vec{\alpha} = (+1, -1, -1, -1, +1)$

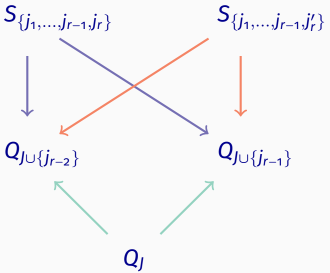
$I \cap J = \{1, 3', 5\}$

QQ-relations at the fork

At the fork node $J = \{j_1, \ldots, j_{r-3}\}$:

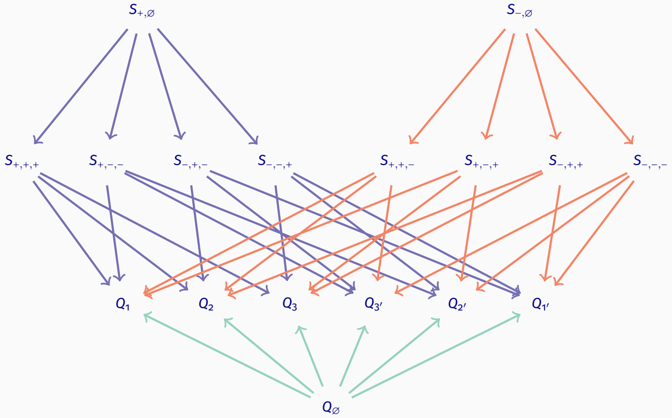
$$Q^{[+1]}_{J\cup\{j_{r-2}\}} Q^{[-1]}_{J\cup\{j_{r-1}\}} - Q^{[-1]}_{J\cup\{j_{r-2}\}} Q^{[+1]}_{J\cup\{j_{r-1}\}} = \frac{\mathcal{T}_{j_{r-2}} - \mathcal{T}_{j_{r-1}}}{\sqrt{\mathcal{T}_{j_{r-2}}\mathcal{T}_{j_{r-1}}}} Q_J S_{\{j_1,\ldots,j_{r-1},j_r\}} S_{\{j_1,\ldots,j_{r-1},j'_r\}} \,,$$

with $S_{\{j_1,\ldots,j_{r-1},j_r\}}$ and $S_{\{j_1,\ldots,j_{r-1},j'_r\}}$ at different spinor nodes



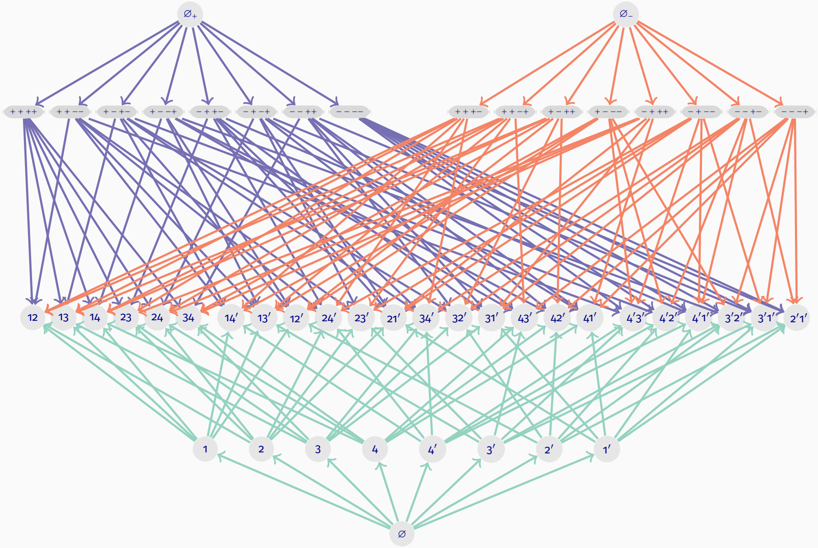
Hasse diagram for so(6)

$A_3 \simeq D_3$



Hasse diagram for so(8)

D_4



Check of QQ-relations

Proof of QQ-relations difficult as we don't have Q-operators for the internal nodes!

Consistency checks possible at fork node for small N:

Compare Q-functions in terms of single index one's

$$Q_{\{i_1,\ldots,i_k\}} = \frac{\left(\sqrt{\tau_{i_1}\cdots\tau_{i_k}}\right)^{k-1}}{\prod_{1\leqslant a<b\leqslant k}\left(\tau_{i_a}-\tau_{i_b}\right)} \frac{\left|Q_{\{i_a\}}^{[k+1-2b]}\right|_k}{\prod_{l=1}^{k-1} Q_{\varnothing}^{[k-2l]}}$$

at $k = r - 2$ and

$$S_I^{[+1]} S_J^{[-1]} - S_I^{[-1]} S_J^{[+1]} = \frac{\tau_{i_{r-1}}\tau_{i_r} - 1}{\sqrt{\tau_{i_{r-1}}\tau_{i_r}}} Q_{I\cap J}$$

Weyl-type formulas

QQ-relations allow to express fundamental transfer in terms of first level Q-operators

Fundamental transfer matrix

$$T_{(1,0)} = Q_{\varnothing}^{[r-1]} Q_{\varnothing}^{[3-r]} \frac{|Q_{\{i_a\}}^{[r+2-2b-2\delta_{b,r}]}|_r}{|Q_{\{i_a\}}^{[r+2-2b]}|_r} + Q_{\varnothing}^{[1-r]} Q_{\varnothing}^{[r-3]} \frac{|Q_{\{i_a\}}^{[2b-r-2+2\delta_{b,r}]}|_r}{|Q_{\{i_a\}}^{[2b-r-2]}|_r}$$

Involves r Q-functions of the first level

In total 2^r equations

Analog of determinant expression in A-type

Weyl-type formulas

Symmetric case

$$T_{(1,s)} = \sum_{l=0}^{s} Q_{\varnothing}^{[2l+r-s-2]} Q_{\varnothing}^{[2+2l-r-s]} \frac{\left| Q_{\{i_a\}}^{[2l+r-s+1-2b+2(s-l)\delta_{b,1}-2l\delta_{b,r}]} \right|_r}{\left| Q_{\{i_a\}}^{[2l+r-s+1-2b]} \right|_r}$$

$T_{(a,s)}$ can be obtained from Hirota equations but we did not find compact expressions

QQ'-type formulas

Bilinear expressions for transfer matrix with $2r$ Q-functions

[Ekhammar,Shu,Volin],[Ferrando,RF,Kazakov]

Extended QQ-system

$$T_{(1,s)}(x) = \sum_{i=1}^{r}\left[\prod_{j\neq i}\frac{\tau_i}{\left(\tau_i-\tau_{j'}\right)\left(\tau_i-\tau_j\right)}\right]\left(Q_{\{i\}}^{[s+r-1]}Q_{\{i'\}}^{[1-r-s]}+Q_{\{i\}}^{[1-r-s]}Q_{\{i'\}}^{[s+r-1]}\right)$$

Extended QQ-system is redundant for spin chain

Reduce number of Q-functions: $T_{(1,s)}=0$ for $-r<s<0$

[Ferrando,RF,Kazakov] $\hookrightarrow$Recover Weyl-type formula

Similar equations for spinorial transfer matrices $T_{(\pm,s)}$

$T_{(a,s)}$ from the Hirota equations

Hirota equations yields $T_{(a,s)}$ in terms of $2r$ Q-functions

$$T_{(a,s)} = \frac{1}{\prod_{k=1}^{a-1} Q_{\varnothing}^{[r+s+2k-a-1]} Q_{\varnothing}^{[1+a-r-s-2k]}} \sum_{1\leqslant i_1<\ldots<i_a\leqslant 2r} h_{i_1}\cdots h_{i_a}\, \mathcal{Q}_{i_1,\ldots,i_a}^{[s+r-1]}\, \mathcal{Q}_{i'_1,\ldots,i'_a}^{[1-s-r]}\,.$$

with

$$h_i = \left[\prod_{j\neq i} \frac{\tau_i}{\left(\tau_i-\tau_{j'}\right)\left(\tau_i-\tau_j\right)}\right]$$

and

$$\mathcal{Q}_{i_1,\ldots,i_k} := \left|Q_{\{i_a\}}^{[k+1-2b]}\right|_k\,.$$

Extensions

BC-type

Similar story for BC-type spin chains [work in progress]

Q-operator construction for gray nodes [RF,Tsymbaliuk]

B_r :

C_r :

Solutions with right asymptotics for white nodes can be obtained from BFN but no trace prescription known.

Q-operators from BFN

Q-operators from BFN

Lax matrices from Drinfeld's current realisation

[Bravermann,Finkelberg,Nakajima] [Kamnitzer,Webster,Weekes,Yacobi][Gerasimov,Kharchev,Lebedev,Oblezin]

$$E_i(x) = -\sum_{r=1}^{a_i} \frac{\prod_{s=1}^{a_{i-1}}(p_{i,r} - p_{i-1,s} - 1)}{(x - p_{i,r})\prod_{s\neq r}(p_{i,r} - p_{i,s})} \mathcal{Z}_i(p_{i,r}) e^{q_{i,r}}$$

$$F_i(x) = \sum_{r=1}^{a_i} \frac{\prod_{s=1}^{a_{i+1}}(p_{i,r} - p_{i-1,s} + 1)}{(x - p_{i,r} - 1)\prod_{s\neq r}(p_{i,r} - p_{i,s})} e^{-q_{i,r}}$$

$$G_i(x) = \frac{\prod_{s=1}^{a_i}(x - p_{i,s})}{\prod_{s=1}^{a_{i-1}}(x - p_{i,s} - 1)} \mathcal{Z}_1(x)\cdots\mathcal{Z}_{i-1}(x)$$

with $[p, e^{\pm q}] = \pm e^{\pm q}$ and $\mathcal{Z}_i(x) = \prod_k (x - x_{i,k})$

Yield Lax matrices at any order of spectral parameter labelled by two partitions λ and μ with $|\lambda| + |\mu| = kr$ and $k \in \mathbb{N}$

[RF,Pestun,Tsymbaliuk]

Q-operators from BFN

Linear solutions for $su(2)$ case

$$L_{XXX}(z) = \begin{pmatrix} z - p & e^{q} \\ -e^{-q}(x_1 - p)(x_2 - p) & z + p + 1 - x_1 - x_2 \end{pmatrix}$$

Highest/lowest weight states $v_0 = e^{x_1 q}$ and $w_0 = e^{x_2 q}$

$$L_{DST}(z) = \begin{pmatrix} z - p & e^{q} \\ e^{-q}(x_1 - p) & 1 \end{pmatrix}$$

Highest/lowest weight states $v_0 = e^{x_1 q}$

$$L_{Toda}(z) = \begin{pmatrix} z - p & e^{q} \\ e^{-q} & 0 \end{pmatrix}$$

No highest/lowest weight state

Q-operators from BFN

Q-operators from L_{DST}

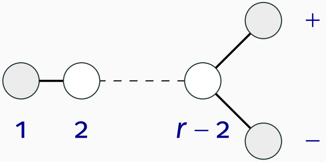
$$Q(z) = \operatorname{tr} \tau_1^{L_{11}(z)} L(z) \otimes \ldots \otimes L(z)$$

with

$$\operatorname{tr} X = \frac{1}{2\pi i} \int_{-\pi i}^{+\pi i} dq \sum_{k=0}^{\infty} e^{-q(x_1-k)} x e^{q(x_1-k)} \, .$$

Lax matrices can be evaluated for BCD-type [RF,Tsymbaliuk]

D_r :



Solutions exist with right asymtotics for white nodes but trace prescription not known!

Outlook

Outlook

D-type QQ-system

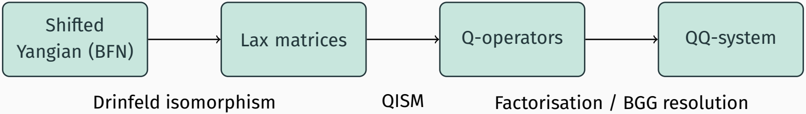
- Proof of QQ-relations and QQ'-relation
- Construction of Q-operators for interior nodes
- Quantum spectral determinant?
- Co-derivative method of [Kazakov,Vieira; Leurent, Kazakov, Tsuboi]
- Relation to Q-operator construction by [Hernandez, Frenkel]
- Open spin chains [RF,Szecsenyi][Basheilac,Tsuboi][Vlaar,Weston]
- Non-compact case and QQ-system for Fishnet [Kazakov et al]

Generalisations [work in progress]

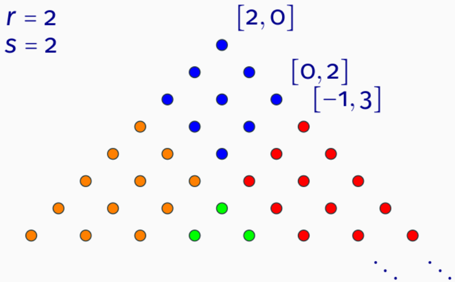
- Generalisation to BC
- Supersymmetric and exceptional Lie algebras work in progress
- Relation to QSC for AdS_4/CFT_3 [Bombardelli,Cavagli,Conti,Fioravanti,Gromov,Tateo]

Conclusion

Universal construction of Q-operators and QQ-systems



Subtraction for D_2



$$\pi_{[s,0]} = \pi^+_{[s,0]} + \pi^+_{[-s-2,0]} - \pi^+_{[-1,s+1]} - \pi^+_{[-1,-s-1]}$$