Q-functions and ’t Hooft operators in 4d Chern-Simons theory

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Outline

- Review: 4d CS theory offers novel perspective on integrability
- Review: Wilson lines give “finite-dimensional” transfer matrices
- Novel: ’t Hooft lines give Q-operators
    - TQ relations from Witten effect
    - Connections to quantum Coulomb branches

4d CS theory

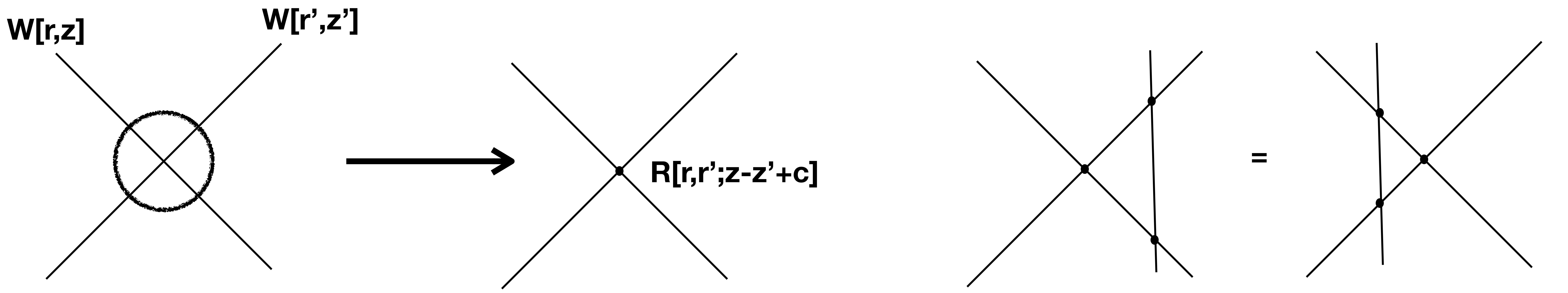
- Fields: $A_{x^1}, A_{x^2}, A_{\bar{z}}$
- Action: $\frac{1}{\hbar}\int dz \wedge \omega_{CS}(A) = \int \frac{z}{\hbar}\mathrm{Tr} F \wedge F$
- Holomorphic-Topological!
- Defined (perturbatively) as a contour path integral on complex fields
- 6d SYM on Omega background, UV completed in String/M-theory

Wilson lines

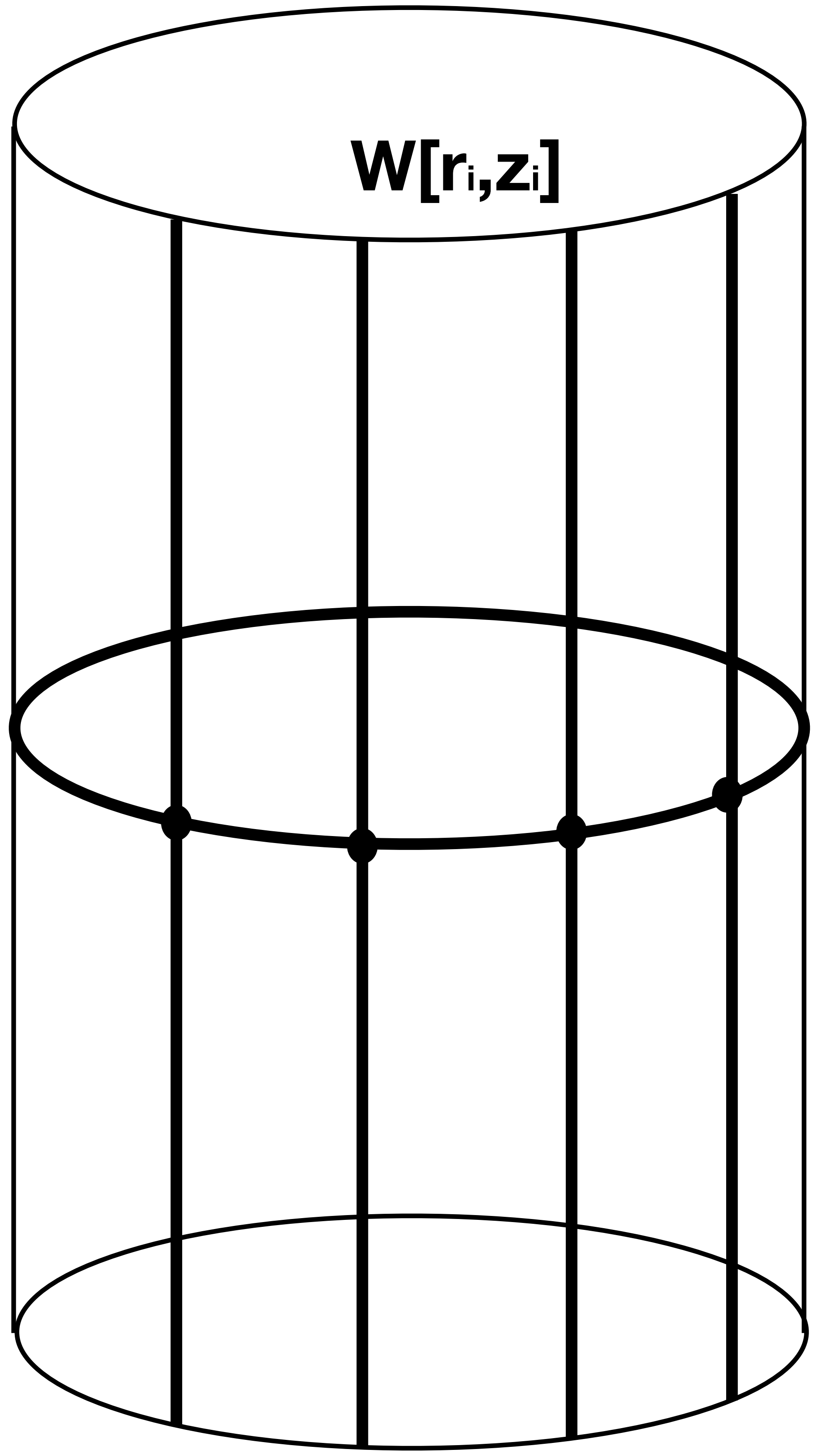
- Naive Wilson loop $\mathrm{Pexp} \int A_1^a(z) t_a dx^1$ has quantum gauge anomalies.
- Fixed by higher counterterms: $\mathrm{Pexp} \int \sum_{n=0}^{\infty} \partial_z^n A_1^a(z) t_a[n] dx^1$
- Extra generators $t_a[n]$ must represent the Yangian!
- Holomorphic-topological, up to a framing anomaly: z shifted by bending

R-matrices

- Gauge interaction propagate only along holomorphic plane.
    - Integrating away gauge fields give interactions local in topological plane
- Wilson line ``crossings” give R-matrices associated to Yangian.



Engineering spin chain transfer matrices



= $T_r[z] = \mathrm{Tr}_r \left[\bigotimes_i R[r, r_i; z - z_i - c] \right] \in \mathrm{End}(\otimes_i r_i)$

Can and usually will include extra twist g from holonomy around the circle

Universal R-matrix and exotic lines

- Path integral for intersection of Wilson lines only depends on t_n and t'_n
  - Universal R-matrix in Yangian x Yangian
- Any Y -> A algebra morphism for A with a trace gives a transfer matrix!
- Example: Y[sl(n)] -> U[sl(n)]/center
  - Depends on spectral parameter and value of Casimir: n parameters
  - $T[z_1,..,z_n]$ behaves as a product of simpler blocks: ``Q-functions”

On Q operators

- Functions whose zeroes are Bethe roots
- Solutions of a recursion relation $Q_{\pm}(z+h) + Q_{\pm}(z-h) = T_{\frac{1}{2}}(z) Q_{\pm}(z)$
- Building blocks for all transfer matrices

$$T_j(z) = Q_+\left(z + (j+\frac{1}{2})h\right) Q_-\left(z - (j+\frac{1}{2})h\right) - Q_-\left(z + (j+\frac{1}{2})h\right) Q_+\left(z - (j+\frac{1}{2})h\right)$$

$$T(z_1, z_2) = Q_+(z_1) Q_-(z_2)$$

- Q operators do NOT come from Yangian representations.
    - Can be defined though Shifted Yangian algebras

‘t Hooft lines

- Disorder defects: prescribe singularity in the fields
- Abelian ’t Hooft loop: prescribe magnetic flux on a sphere around defect
- Non-Abelian: use Abelian singular solution as a reference
  - Naive data: magnetic weight, U(1)->G

‘t Hooft operators II

- Precise definition subtle and theory-dependent:
  - Define good phase space of singular gauge field configurations
  - Quantize it
- 4d CS theory: direct definition but also inheritance from 6d SYM and String/M-theory
  - Claim: ’t Hooft operators give Q-operators

’t Hooft-Wilson operators

- A line defect can have both magnetic and electric charges
  - Magnetic charge breaks G to H, can dress by Wilson line for H
  - Witten effect: Dyon charges shifted by theta angle to e + theta/(2pi) m
  - Theta angle z: D(m,e)[z] = D(m,e+m)[z+h]
- Fusion relations analogous to TQ relations
  - SL(2): W[z] H[z]=D(1,1)[z]+D(1,-1)[z]=H[z+h] + H[z-h]

Quantization of BPS moduli spaces

- Phase space, as complex manifold = solutions of Bogomolny equations
  - Well studied, depends on ’t Hooft charges and flux at infinity.
- Use 3d N=4 in Omega background to define defects in UV:
  - Phase space = Coulomb branch of ADE quivers
  - Quantization from Omega deformation.
  - Naturally equipped with map from (Shifted) Yangians, as desired.

Perfect match

- Collection of ’t Hooft charges at positions z_i=> quantum Coulomb branch algebra A[z_i]
- Y->A[z_i] and trace on A[z_i] => transfer matrix T[z_i]
- T[z_i] = product of Q_i[z_i]
- QQ and TQ relations from Category O of A[z_i]

Reverse application

- Supersymmetric sphere partition function of 3d N=4 quivers is a (special) twisted trace on quantum Coulomb branch
- Express protected correlators as transfer matrices of a spin chain
- Applications to interfaces in 4d N=4 SYM