Lattice nonlinear Schrödinger: history and open problems

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The Hamiltonian is:

$$H = \int dx(\partial_x \psi^\dagger \partial_x \psi + \kappa \psi^\dagger \psi^\dagger \psi \psi), \qquad \kappa \geq 0$$

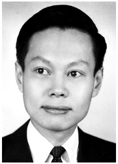
$$\{\psi(x), \psi^\dagger(y)\} = i\delta(x-y)$$

It is integrable also in classical case. It has infinitely many conservation laws and Lax representation. A.Takhtajan and L. Faddeev, *"Hamiltonian Methods in the Theory of Solitons"*, *Springer.*

$$\partial_t L_n = M_{n+1}(\lambda) L_n(\lambda) - L_n(\lambda) M_n(\lambda)$$

An integer n is a discrete space variable $x = n\Delta$ and Δ is lattice spacing.

R matrix



We shall discuss the quantum case. Lax representation follows from Yang-Baxter equation. https://arxiv.org/pdf/0910.0295.pdf

$$R(\lambda,\mu)\left(L_n(\lambda)\bigotimes L_n(\mu)\right)=\left(L_n(\mu)\bigotimes L_n(\lambda)\right)R(\lambda,\mu)$$

The $R(\lambda,\mu)$ solves the Yang-Baxter equation: C.N. Yang. PRL 19, (1967)
https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.19.1312
C.N. Yang found the following solution:

$$R(\lambda,\mu)=i\mathbb{1}+(\mu-\lambda)\Pi$$

This is the same R matrix, which describes XXX Heisenberg chain with spin 1/2 . Here Π is permutation. C.N. Yang Phys. Rev. 168, (1968) 1920
https://journals.aps.org/pr/abstract/10.1103/PhysRev.168.1920

Note

Note that for $\lambda-\mu=i$ the $R(\lambda,\mu)=i\left(\mathbb{1}-\Pi\right)$ is degenerate. It is 1D projector.

QISM formulation

Sklyanin, Takhtajan and Faddeev formulated algebraic Bethe Ansatz and embedded the model into quantum inverse scattering method. An approximate L operator on a dense lattice is:

$$L_n(\lambda) = \begin{pmatrix} 1 - \frac{i\lambda\Delta}{2} & -i\sqrt{\kappa}\chi_n^\dagger \\ i\sqrt{\kappa}\chi_n & 1 + \frac{i\lambda\Delta}{2} \end{pmatrix} + O(\Delta^2), \qquad \chi = \psi\Delta$$

An integer n is a discrete space variable $x = n\Delta$,
and Δ is lattice spacing. The χ_n is the quantum field:

$$[\chi_n, \chi_m^\dagger] = \Delta\, \delta_{nm}$$

in the limit $\Delta \to 0$. The λ is the spectral parameter and κ is a coupling constant.

Exact lattice Lax operator (all orders in Δ) was constructed by A. Izergin and V. Korepin in Doklady Akademii Nauk, 1981
https://arxiv.org/pdf/0910.0295.pdf
see also Nuclear Physics B 205 [FS5], 401, 1982

$$L_j(\lambda) = \begin{pmatrix} 1 - \frac{i\lambda\Delta}{2} + \frac{\kappa}{2}\chi_j^\dagger\chi_j & -i\sqrt{\kappa}\chi_j^\dagger\varrho_j \\ i\sqrt{\kappa}\varrho_j\chi_j & 1 + \frac{i\lambda\Delta}{2} + \frac{\kappa}{2}\chi_j^\dagger\chi_j \end{pmatrix}.$$

$$[\chi_j, \chi_l^\dagger] = \Delta\delta_{j,l} \quad \text{and} \quad \varrho_j = (1 + \frac{\kappa}{4}\chi_j^\dagger\chi_j)^{\frac{1}{2}},$$

here $\kappa > 0$, and $\Delta > 0$. The same R matrix.

We can rewrite the L operator as XXX Heisenberg chain
Zeitschrift für Physik 49, (1928) 619-636
https://link.springer.com/article/10.1007/BF01328601:

$$L_j^{XXX} = -\sigma^z L_j = i\lambda + S_j^k \otimes \sigma^k$$

$$S_j{}^+ = -i\sqrt{\kappa}\chi_j^\dagger \varrho_j, \quad S_j{}^- = i\sqrt{\kappa}\varrho_j \chi_j, \quad S_j{}^z = (1 + \frac{\kappa}{2}\chi_j^\dagger \chi_j).$$

The σ are Pauli matrices. The S_j form a representation of $SU(2)$ algebra with negative spin

$$s = -\frac{2}{\kappa\Delta}$$

V. Kazakov and K. Zarembo
https://arxiv.org/pdf/hep-th/0410105.pdf
N. Gromov and V. Kazakov
https://arxiv.org/pdf/hep-th/0510194.pdf

Antipode and quantum determinant

The monodromy matrix and transfer matrix:

$$T_0(\lambda) \equiv L_{0\mathcal{L}}(\lambda) \dots L_{01}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_0, \qquad \tau(\lambda) = tr_0\, T_0(\lambda).$$

$R(\lambda,\mu)\,(T_0(\lambda) \bigotimes T_0(\mu)) = (T_0(\mu) \bigotimes T_0(\lambda))\,R(\lambda,\mu)$

The antipode of quantum monodromy matrix is:

$$T(\lambda)^{-1} = {d_q}^{-L}(\lambda)\sigma^y T^t(\lambda+i)\sigma^y$$

$$d_q(\lambda) = \Delta^2(\lambda-\nu)(\lambda-\nu+i)/4 \qquad \nu = -2i/\Delta$$

This is a deformation of the Cramer's formula

http://pi.math.cornell.edu/~andreim/Lec17.pdf

The difference is a shift of the spectral parameter by i.

The denominator is the *quantum determinant*:

$\det_q T(\lambda) = A(\lambda)D(\lambda+i) - B(\lambda)C(\lambda+i) = {d_q}^L(\lambda)$

This was discovered in 1981 in https://arxiv.org/pdf/0910.0295.pdf

Remark

When $\lambda - \mu = i$ the R matrix turns into one dimensional projector.

Hamiltonian of LNS was proposed by Tarasov, Takhtadzhyan and Faddeev. It describes interaction of nearest neighbors:

$$\mathcal{H} = \sum_{k=1}^{L} H_{k,k+1}$$

The density of the Hamiltonian is expressed in terms of $J_{k,k+1}(J_{k,k+1}+1) = 2\vec{S}_k \otimes \vec{S}_{k+1}$

$$H_{k,k+1} = -\psi(-J_{k,k+1}) - \psi(J_{k,k+1}+1) + 2\psi(1); \quad \psi(x) = d \ln \Gamma(x)/dx$$

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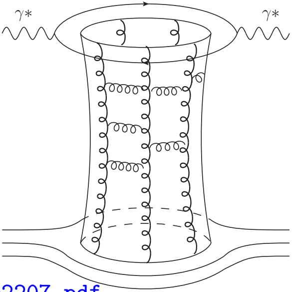
It has application to high energy physics: describes deep inelastic scattering of an electron on a nucleon
https://en.wikipedia.org/wiki/Deep_inelastic_scattering

QCD describes deep inelastic scattering. Lev Lipatov considered small Bjorken x
http://www.scholarpedia.org/article/Bjorken_scaling.
At high energy, the scattering amplitudes are described by the exchange of gluons dressed by virtual gluon loops: so-called Reggeized gluons. In the limit of large number of colors Nc (with fixed $g^2 N_c$, where g is the QCD coupling), the corresponding Feynman diagrams have the topology of the cylinder. The Hamiltonian describing the interactions of Reggeized gluons reduces to the sum of terms describing the pairwise near-neighbor interactions: the XXX spin chain
https://arxiv.org/pdf/hep-th/9311037.pdf



The scattering of a lepton on a hadron is a sum of Feynman diagrams. In leading logarithmic approximation ladder diagrams dominate. Quarks exchange gluons. The Hamiltonian of LNS describes interactions of the gluons.

Faddeev, Korchemsky
https://arxiv.org/pdf/hep-th/9404173.pdf

Alvarez-Gaume,
https://arxiv.org/pdf/0804.1464.pdf

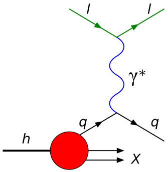
Zarembo
https://arxiv.org/pdf/hep-th/0411191.pdf

Kazakov, Marshakov https://arxiv.org/pdf/hep-th/0402207.pdf

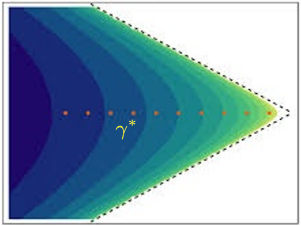
Our paper: https://arxiv.org/pdf/1909.00800.pdf

Virtual photon in semi-classical approximation

The electron emits a virtual photon γ^*, which cuts thought the hadron.



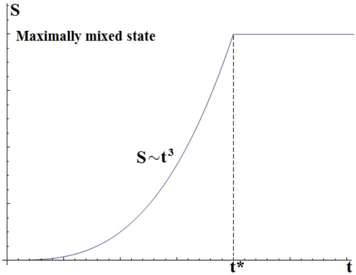
A signal spreads through distribution of quarks and gluons in the hadron with a velocity smaller than the speed of light. The size of the γ^*, is much smaller than the size of the baryon. The photon cause a shock wave inside the baryon: it changes the density of entropy. The volume of the cone of Cherenkov radiation is cubic in time.



Entropy evolution

Open problem:

Description of time evolution of entropy, while the virtual photon cuts through the hadron. Is the growth cubic in time?



Note

Experiment: Data needs to be analyzed from DESY's electron-proton collider HERA.
https://cerncourier.com/a/the-most-precise-picture-of-the-proton/

Let us return back to LNS. It is equivalent to XXX chain with spin $s = -1$
The Bethe equations are

$$\left(\frac{\lambda_k + is}{\lambda_k - is}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{N} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i} \xrightarrow{s=-1} \left(\frac{\lambda_k - i}{\lambda_k + i}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{N} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}$$

$$k = 1, \cdots, N$$

These are periodic boundary conditions. The energy is:

$$E = \sum_{j=1}^{N} \frac{-2}{\lambda_j^2 + 1},$$

Theorem 1. If solutions of Bethe equations (for $s = -1$) exist then they are **real numbers**.

$$\left(\frac{\lambda_k - i}{\lambda_k + i}\right)^L = \prod_{j=1,\ j\neq k}^{N} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}, \qquad k = 1, \cdots, N.$$

Proof: Let us use the following properties:

$$\text{LHS:} \left|\frac{\lambda - i}{\lambda + i}\right| \leq 1, \quad \text{when } \mathrm{Im}\lambda \geq 0; \quad \left|\frac{\lambda - i}{\lambda + i}\right| \geq 1, \quad \text{when } \mathrm{Im}\lambda \leq 0;$$

$$\text{RHS:} \left|\frac{\lambda + i}{\lambda - i}\right| \geq 1, \quad \text{when } \mathrm{Im}\lambda \geq 0; \quad \left|\frac{\lambda + i}{\lambda - i}\right| \leq 1, \quad \text{when } \mathrm{Im}\lambda \leq 0;$$

If we denote the one with maximal imaginary part as $\lambda_{max} \in \{\lambda_j\}$, then

$$\mathrm{Im}\ \lambda_{max} \geq \mathrm{Im}\ \lambda_j, \quad j = 1, \cdots, N.$$

For $\lambda_k = \lambda_{max}$

$$\left|\frac{\lambda_{max} - i}{\lambda_{max} + i}\right|^L = \left|\prod_{j=1}^{N} \frac{\lambda_{max} - \lambda_j + i}{\lambda_{max} - \lambda_j - i}\right| \geq 1.$$

Due to LHS, this results in: $\mathrm{Im}\ \lambda_j \leq \mathrm{Im}\ \lambda_{max} \leq 0$
Similarly, one has $0 \leq \mathrm{Im}\ \lambda_{min} \leq \mathrm{Im}\ \lambda_j \to$ so $\mathrm{Im}\ \lambda_j = 0$

The logarithm of **Bethe equations** for the model $s = -1$,

$$2\pi n_k = \sum_{j=1}^{N} \theta(\lambda_k - \lambda_j) + L\,\theta\,(\lambda_k)\,,$$

Here n_k are different integer (or half integer) numbers: Pauli principle
http://insti.physics.sunysb.edu/~korepin/PDF_files/Pauli.pdf

$$\theta(\lambda) = -\theta(-\lambda) = i \ln\left(\frac{i\kappa + \lambda}{i\kappa - \lambda}\right); \quad -\pi < \theta(\lambda) < \pi, \quad \mathrm{Im}\,\lambda = 0$$

$$\theta'(\lambda - \mu) = K(\lambda, \mu) = \frac{2\kappa}{\kappa^2 + (\lambda - \mu)^2}, \qquad K(\lambda) = K(\lambda, 0).$$

All λ_k are also different.

Yang's action is convex

Theorem 2. The solutions of the logarithmic form **Bethe equations** exist. Logarithmic Bethe equations are the extremums of the Yang action:

$$S = L \sum_{k=1}^{N} \theta_1(\lambda_k) + \frac{1}{2} \sum_{k,j}^{N} \theta_1(\lambda_k - \lambda_j) - 2\pi \sum_{k=1}^{N} n_k \lambda_k,$$

$\theta_1(\lambda) = \int_0^\lambda \theta(\mu) d\mu$. Bethe equations: $\partial S / \partial \lambda_j = 0$.

$$\frac{\partial^2 S}{\partial \lambda_j \partial \lambda_l} = \delta_{jl} [L\, K(\lambda_j) + \sum_{m=1}^{N} K(\lambda_j, \lambda_m)] - K(\lambda_j, \lambda_l)$$

Consider some real vector v_j . The quadratic form is positive:

$$\sum_{j,l} \frac{\partial^2 S}{\partial \lambda_j \partial \lambda_l} v_j v_l = \sum_{j=1}^{N} L\, K(\lambda_j) v_j^2 + \sum_{j>l}^{N} K(\lambda_j, \lambda_l)(v_j - v_l)^2 \geq 0$$

The $K(\lambda_j)$ are positive. The action is convex: it has unique minimum. Solution of Bethe equation exists and unique [in the logarithmic form].

Side remark

The same second derivative appears later in the theory.
The square of the norm of the Bethe wave function is a determinant:

$$\langle \Phi_N | \Phi_N \rangle = \det \left(\frac{\partial^2 S}{\partial \lambda_j \partial \lambda_l} \right)$$

Similar formula was conjectured by M. Gaudin for the continuous case of NS. The formula was proved by V. Korepin in 1982.
The proof also works on the lattice.
http://insti.physics.sunysb.edu/~korepin/PDF_files/norm.PDF.

$$\frac{\partial^2 S}{\partial \lambda_j \partial \lambda_l} = \delta_{jl}[L\, K(\lambda_j) + \sum_{m=1}^{N} K(\lambda_j, \lambda_m)] - K(\lambda_j, \lambda_l)$$

The thermodynamic limit at zero temperature

For positive κ all λ_j has to be different: Pauli principle in the momentum space is valid.
A. Izergin and V.Korepin; Letters in Mathematical Physics 1982:
http://insti.physics.sunysb.edu/~korepin/PDF_files/Pauli.pdf
In the limit $L \to \infty$ and $N \to \infty$, the λ_j are condensed into Fermi sphere $[-q, q]$.
The distribution function $\rho_p(\lambda_j) = \frac{1}{L(\lambda_{j+1} - \lambda_j)}$ satisfy:

$$2\pi \rho_p(\lambda) = \int_{-q}^{q} K(\lambda, \mu) \rho_p(\mu) d\mu + K(\lambda)$$

$$K(\lambda, \mu) = \frac{2\kappa}{\kappa^2 + (\lambda - \mu)^2}, \quad K(\lambda) = K(\lambda, 0), \quad \int_{-q}^{q} \rho_p(\lambda) d\lambda = D = \frac{N}{L}$$

For $\kappa = 0$ Fermi sphere collapse: the ground state is Bose-Einstein condensate.

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Open problem:

Is to analyze the integral equation in the limit of $\kappa \to 0$. Describe singularities.
The $K(\lambda, \mu) \to 2\pi\delta(\lambda - \mu)$: the integral cancel the LHS ...?

Collapse of the Fermi sphere in the continuous case

In the continuous case of nonlinear Schrödinger the integral equation is:

$$2\pi\rho_p(\lambda) = \int_{-q}^{q} K(\lambda,\mu)\rho_p(\mu)d\mu + 1.$$

This is Lieb-Liniger equation. In the limit $\kappa \to 0$ the $K(\lambda,\mu) \to 2\pi\delta(\lambda-\mu)$.The integral cancel the LHS. The limit was studied by
S. Prolhac https://arxiv.org/pdf/1610.08912.pdf
G. Lang https://arxiv.org/pdf/1907.04410.pdf
C. Tracy, H. Widom https://arxiv.org/pdf/1609.07793.pdf

The decomposition is in $\sqrt{\kappa}$ and $\log\kappa$. Coefficients are objects of number theory.

In the lattice case the limit is an open problem.

Energy of elementary excitation

Return to the lattice case: special case XXX with spin $s = -1$. Considering the grand canonical ensemble, the energy spectrum becomes

$$E_h = \sum_{j=1}^{N} \left(\frac{-2}{\lambda_j^2 + 1} - h \right)$$

The h is the chemical potential. In thermodynamic limit the energy of elementary excitation $\varepsilon(\lambda)$ satisfies the linear integral equation

$$\varepsilon(\lambda) - \frac{1}{2\pi} \int_{-q}^{+q} K(\lambda, \mu) \varepsilon(\mu) d\mu = \frac{-2}{\lambda^2 + 1} - h \equiv \varepsilon_0(\lambda),$$

$$\varepsilon(q) = \varepsilon(-q) = 0$$

Remark

The elementary excitation has a topological charge: it does not fit into periodical boundary conditions, we have to change the boundary conditions into anti-periodic.

Construction of elementary excitation

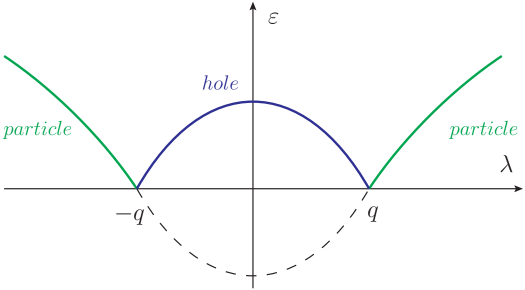

Figure 1: The energy of the elementary excitation as a function of λ.
For $-q < \lambda < q$ elementary excitation is a hole, but it is the particle for other values of λ.

In the infinite volume limit any energy level is a scattering state of several elementary excitations with different momenta.

Momentum of the elementary excitation.

The momentum of the particle $\mathbf{k}(\lambda_p)$ is

$$\mathbf{k}(\lambda_p) = p_0(\lambda_p) + \int_{-q}^{q} \theta(\lambda_p - \mu)\rho_p(\mu) d\mu, \quad \theta(\lambda) = p_0(\lambda) = i \ln\left(\frac{i+\lambda}{i-\lambda}\right).$$

The momentum $\mathbf{k}_h(\lambda_h)$ of elementary hole excitation is

$$\mathbf{k}_h(\lambda_h) = -p_0(\lambda_h) - \int_{-q}^{q} \theta(\lambda_h - \mu)\rho_p(\mu) d\mu.$$

where $-q < \lambda_h < q$. At zero temperature all the observables are described by a linear integral equation.

The scattering matrix of two elementary excitation is a transition coefficient:

$$S = \exp\{-i\phi(\lambda_p, \lambda_h)\},$$

the scattering phase satisfies the integral equation:

$$\phi(\lambda_p, \lambda_h) - \frac{1}{2\pi}\int_{-q}^{+q} K(\lambda_p, \nu)\phi(\nu, \lambda - \lambda_h) d\nu = \theta(\lambda_p - \lambda_h).$$

$$\theta(\lambda) = -\theta(-\lambda) = i \ln\left(\frac{i\kappa + \lambda}{i\kappa - \lambda}\right); \quad -\pi < \theta(\lambda) < \pi, \quad \operatorname{Im}\lambda = 0$$

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Remark

Many body scattering matrix is a product of pairwise scattering matrices. This can be used as a definition of complete integrability in many body quantum mechanics.

The Yang-Yang equation describes quantum thermodynamics of LNS:

$$\varepsilon(\lambda) = \frac{-2}{\lambda^2+1} - h - \frac{T}{2\pi}\int_{-\infty}^{+\infty} K(\lambda,\mu)\ln(1+e^{-\varepsilon(\mu)/T})d\mu,$$

$$\frac{\rho_h(\lambda)}{\rho_p(\lambda)} = e^{\varepsilon(\lambda)/T}, \qquad D = \frac{N}{L} = \int_{-\infty}^{\infty} \rho_p(\lambda)d\lambda.$$

The $\varepsilon(\lambda)$ is the energy of the stable excitation.

Open problem

Analytical solution in limiting case: $\kappa \to 0$. The $K(\lambda,\mu) \to 2\pi\delta(\lambda-\mu)$:

The free energy is:

$$\mathcal{F} = Nh - \frac{LT}{2\pi} \int_{-\infty}^{+\infty} K(\mu) \ln(1 + \exp(-\varepsilon(\mu)/T)) d\mu.$$

The pressure is:

$$\mathcal{P} = -\left(\frac{\partial \mathcal{F}}{\partial L}\right)_T = \frac{T}{2\pi} \int_{-\infty}^{+\infty} K(\mu) \ln(1 + e^{-\varepsilon(\mu)/T}) d\mu.$$

Thermal entropy is:

$$S = -\frac{\partial \mathcal{F}}{\partial T} = \frac{L}{2\pi} \int_{-\infty}^{+\infty} K(\mu) \left[\ln(1 + e^{-\varepsilon(\mu)/T}) + \frac{\varepsilon(\mu)}{T(e^{\varepsilon(\mu)/T} + 1)} \right] d\mu.$$

Thermodynamics of NS was realized experimentally in quantum optics.
It was build in optical lattice by N. J. van Druten
https://arxiv.org/pdf/0709.1899.pdf

At zero temperature the ground state $|gs\rangle$ is unique. The entropy of the ground state is zero. Let us consider a block of x sequential lattice cites. We interpret the rest of the lattice as an environment. We trace away the environment: this gives us the density matrix of the block $\rho = tr_E\left(|gs\rangle\langle gs|\right)$. The von Neumann entropy of the block is a complicated function of x, but for large x it scales logarithmically

$$S_{vN} = -tr(\rho \log \rho) \to \frac{1}{3}\log(x) \quad \text{as} \quad x \to \infty$$

similar to the continuous case of NS.
https://arxiv.org/pdf/cond-mat/0311056.pdf

Remark:
The logarithm is not universal. In some spin chains the entropy scales as a fractional power of x. https://arxiv.org/pdf/1605.03842.pdf

Renyi entropy

The Renyi entropy is defined as

$$S_R = \frac{\ln\left(tr\rho^{\alpha}\right)}{1-\alpha}, \qquad \alpha > 0$$

The α is a new parameter. For LNS
the Renyi entropy also scales logarithmically with the size of a block

$$S \to \frac{(1+\alpha^{-1})\log x}{6}$$

as in XX spin chain https://arxiv.org/pdf/quant-ph/0304108.pdf

Note

In some spin chains the Renyi entropy is not an analytical function of α: it scales differently for different α https://arxiv.org/pdf/1806.04049.pdf
This is Stokes phenomenon https://en.wikipedia.org/wiki/Stokes_phenomenon

Open problem

The LNS out of equilibrium.
Many different ideas.

Mark Mezei used membrane theory to study the entanglement dynamics https://arxiv.org/pdf/1912.11024.pdf
Ryusuke Hamazaki https://arxiv.org/pdf/2012.11822.pdf
Fabian Essler introduced Quench Action. It worked for continiuos NS https://arxiv.org/pdf/2102.09987.pdf
Boundary Conformal Field Theory was used for study of time evolution of the entanglement entropy by Olalla A. Castro-Alvaredo, Mt Lencss, Istvn M. Szcsnyi, Jacopo Viti https://arxiv.org/pdf/1907.11735.pdf

Open problems

Can we calculate correlation functions in XXX with negative spin?
First at zero temperature, time independent in the infinite volume.

At spin $1/2$ correlation functions in XXX chain can be expressed as polynomials [with rational coefficients] of the values of Riemann zeta function with odd arguments
H.E. Boos, V.E. Korepin https://arxiv.org/pdf/hep-th/0104008.pdf
T. Miwa, F. Smirnov https://arxiv.org/pdf/1802.08491.pdf

The values of Riemann zeta function with odd arguments are celebrated object of number theory. They are conjectured to be transcendental numbers, algebraically independent over the field of rational numbers, see wikipedia: Apery's theorem.

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Open problem

Can we describe correlation functions of the XXX with negative spin by number theory?

Note

Dirk Kreimer https://www2.mathematik.hu-berlin.de/~kreimer/

Extra symmetry in the infinite volume is an open problem.

At spin 1/2 the XXX chain gains an additional symmetry in the thermodynamic limit. It is the YANGIAN SYMMETRY (infinite dimensional quantum group):
https://arxiv.org/abs/hep-th/9211133v2
Hubbard model also has Yangian symmetry
https://arxiv.org/pdf/hep-th/9310158.pdf
Also planar N = 4 SYM https://arxiv.org/pdf/1004.5423.pdf
Yangian can be used as a tool for investigation of integrability features of QCD at high energies.

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https://arxiv.org/pdf/hep-th/9310158.pdf
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Yangian can be used as a tool for investigation of integrability features of QCD at high energies.

Open problem

Does an additional symmetry arise in XXX with negative spin in the limit of infinitely long lattice?

Side remark

M. Ablowitz and J. Ladik in J. Math. Phys. 17, (1976) 1011 constructed a different integrable discretization of nonlinear Schoedinger. The L-operator is different.

$$\frac{i}{2}\frac{\partial}{\partial t}\psi(n,t) = (1+4\psi(n,t)\psi^\dagger(n,t))(\psi(n+1,t)+\psi(n-1,t))$$

$$-\frac{i}{2}\frac{\partial}{\partial t}\psi^\dagger(n,t) = (1+4\psi(n,t)\psi^\dagger(n,t))(\psi^\dagger(n+1,t)+\psi^\dagger(n-1,t)).$$

Tim Hoffmann proved that in classical case the L operator of Ablowiz-Ladik version is gauge equivalent to the Izergin-Korepin version of LNS Physics Letters A 265 (2000) 62-67. http://insti.physics.sunysb.edu/~korepin/PDF_files/Hoff.pdf

Note

Classical Ablowitz-Ladik has important applications, see Phys.Rev.Lett. 70 (1993) 1704-1708. http://insti.physics.sunysb.edu/~korepin/PDF_files/ttc.pdf
It describe space, time and temperature dependent correlation function in a spin chain.

In classical case Tim Hoffmann constructed an integrable double discrete version of nonlinear Schoedinger and related it to geometry:
Discrete Hashimoto Surfaces and a Doubly Discrete Smoke-Ring Flow.
Discrete Differential Geometry, (2008), Vol. 38, pp 95-115.
https://link.springer.com/chapter/10.1007%2F978-3-7643-8621-4_5.

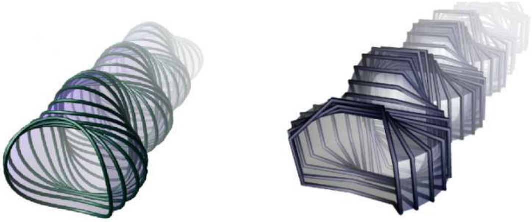

Figure 2: The right figure describes the solutions of the double discrete evolution.

Modern way is to describe them by tropical geometry
https://en.wikipedia.org/wiki/Tropical_geometry

Open problem

The field was developed by Alexander Bobenko and Yuri Suris.
The book: Discrete Differential Geometry, Integrable Structure
https://books.google.com/books/about/Discrete_Differential_Geometry.html?id=H1u10anYfigC
It has applications. No Hamiltonian formulation, just recursion relations.

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Open problem

Quantization of the double discrete nonlinear Schrödinger.

Note

Maybe Taylor series in time step will work? Lowest order corresponds to the continuous time. Higher orders can be restored from the requirement of integrability.
https://stonybrook.zoom.us/rec/share/o9gBCHXv4RZuvAOm16QPvlQlTC9w15Yeb9nzumHysVmrwX58sIJ8-3LSBkvnMrjr.8bp9ud19s266aer6 Passcode: 04P.7j?V

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