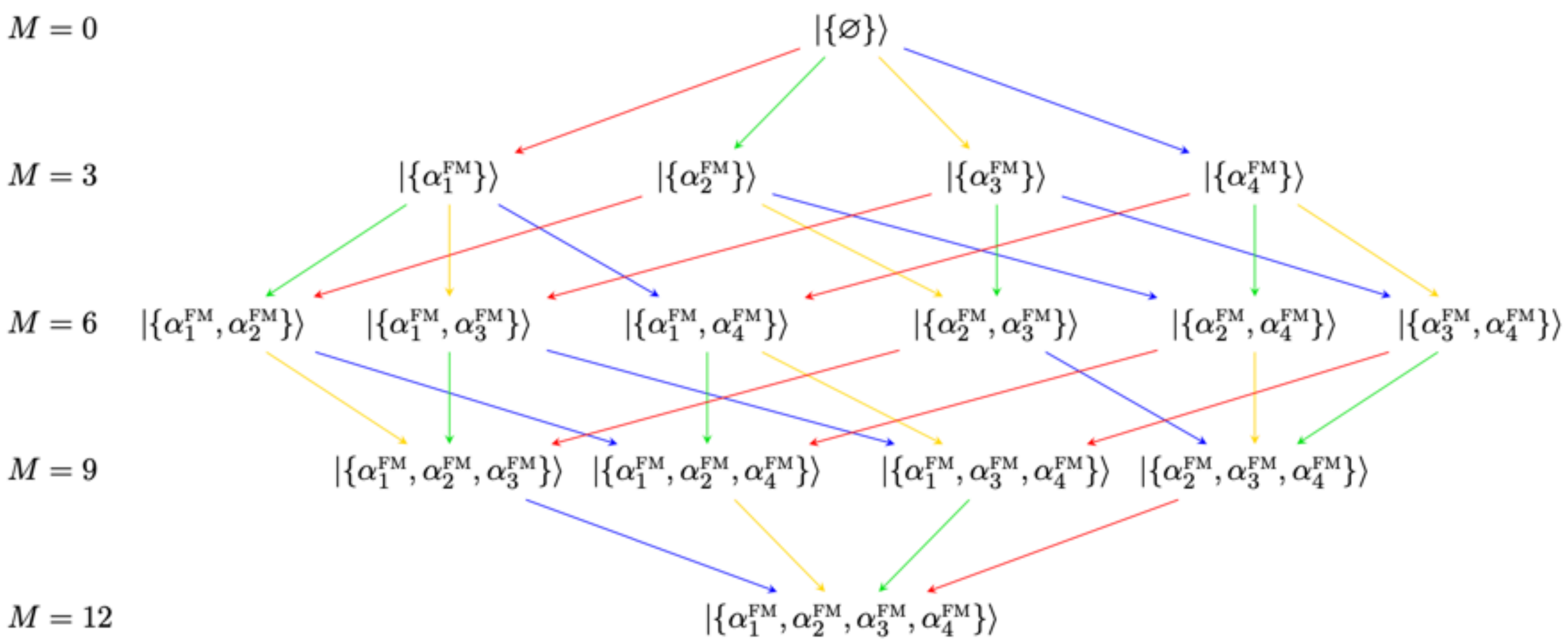
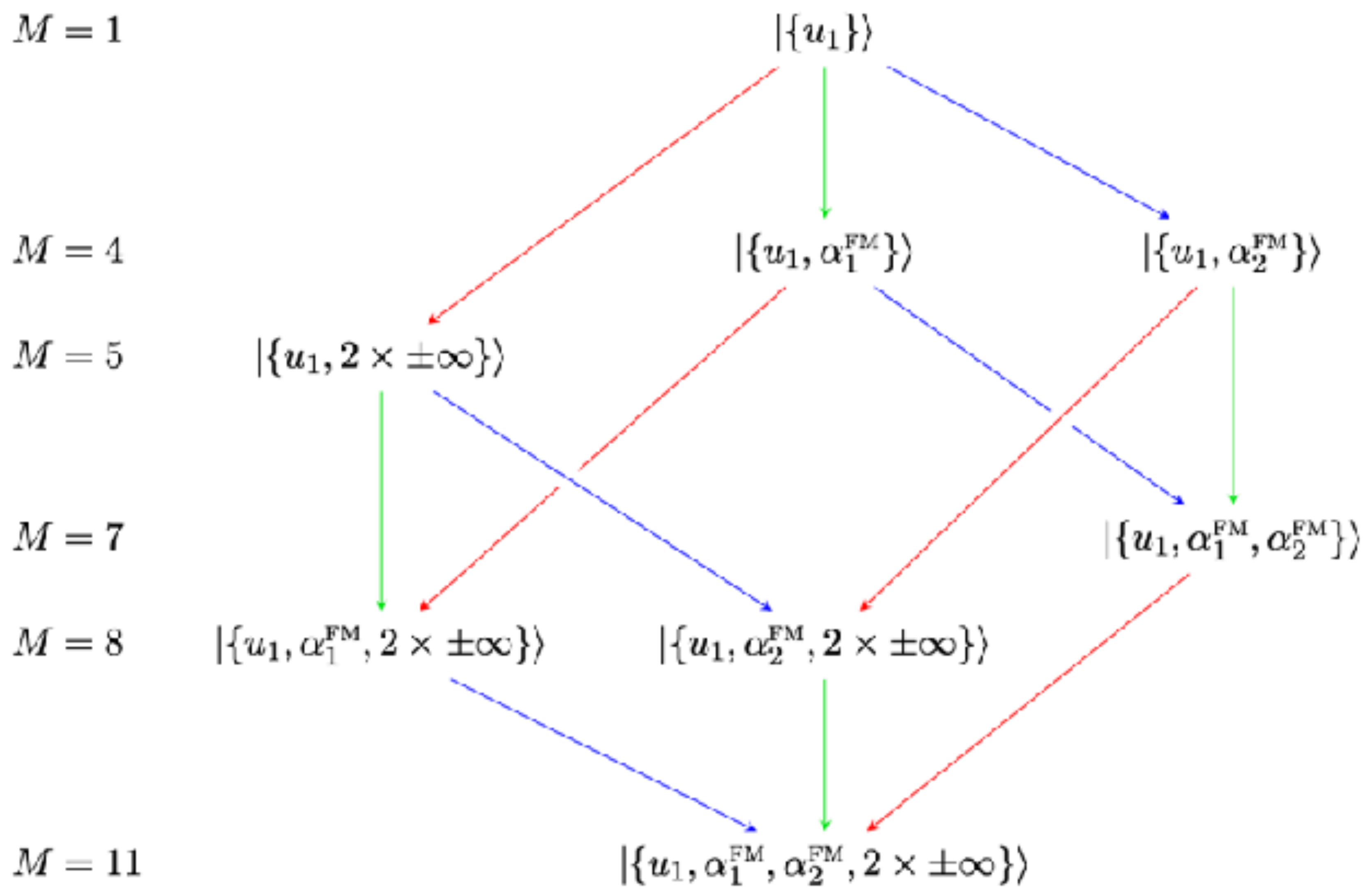
# On the Q operator and spectrum of spin-1/2 XXZ model at root of unity




**arXiv: 2011.xxxxx (to appear)**


with Jean-Sébastien Caux, Jules Lamers and Vincent Pasquier

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# Motivation: why spectrum?

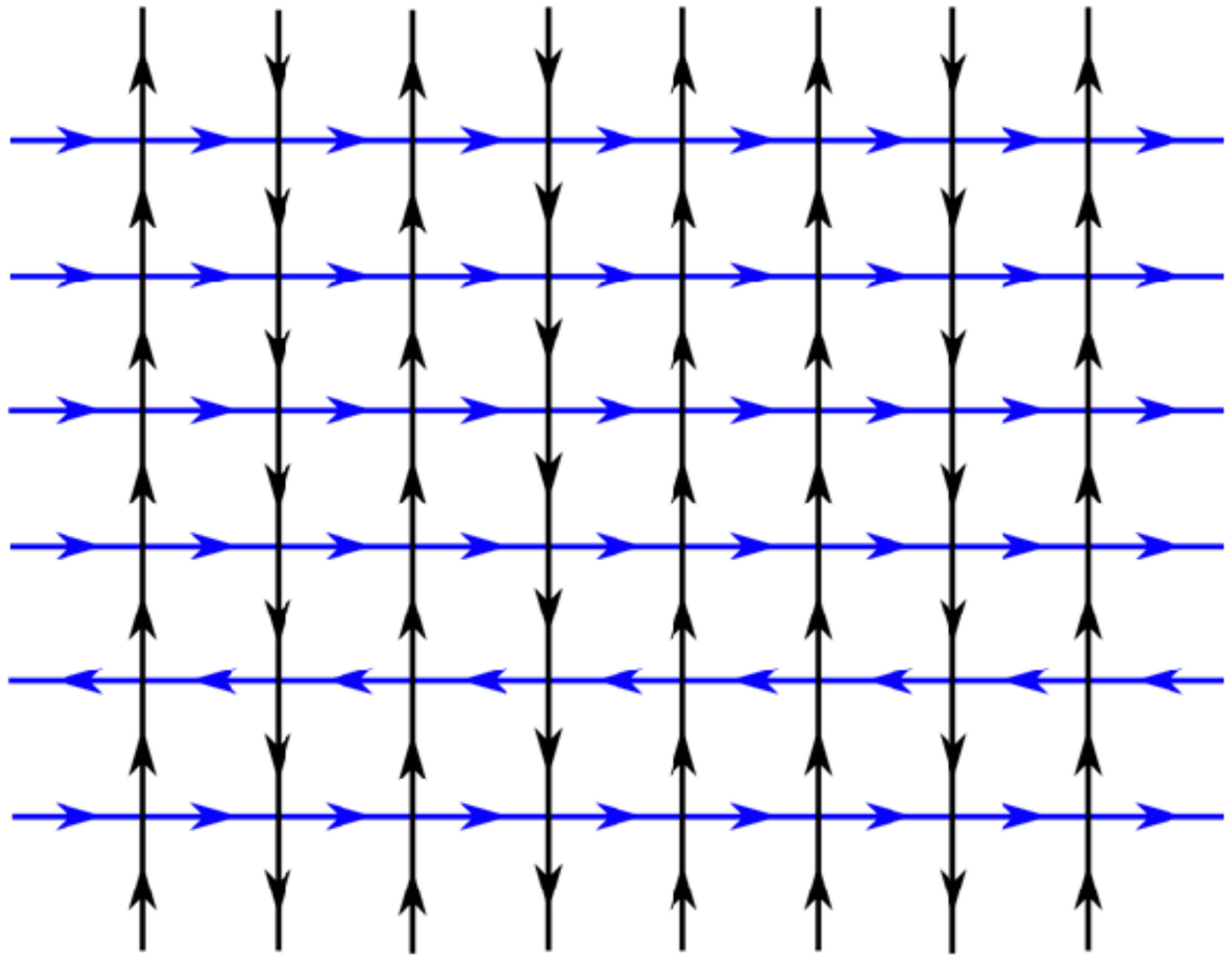
$$\hat{\varrho} = \sum_n c_n |n\rangle\langle n| \quad \Rightarrow$$

entanglement entropy

thermalisation

out-of-equilibrium properties (quantum quenches, hydrodynamics, etc)

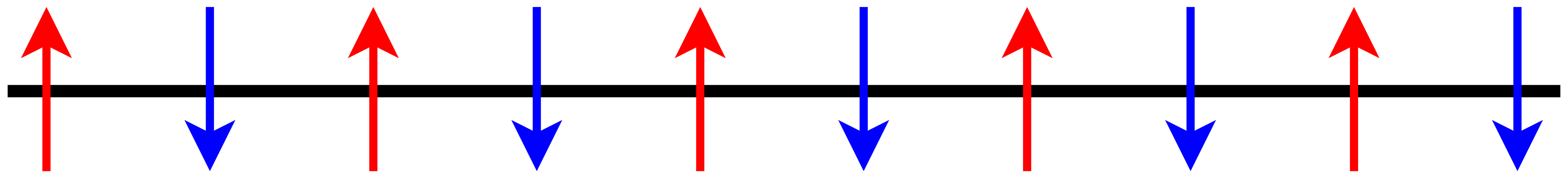
***spectrum***



$$\mathcal{Z} = \mathrm{Tr}\left(T^M(u)\right)$$

partition function
for 6-vertex model

Jacobsen et al, JHEP, 2019
Bajnok et al, JHEP, 2020



Néel quench

$$\langle\psi_0|\hat{O}(t)|\psi_0\rangle = \sum_{n,m} \frac{\langle\psi_0|n\rangle\langle n|\hat{O}|m\rangle\langle m|\psi_0\rangle}{\langle n|n\rangle\langle m|m\rangle} e^{\mathrm{i}(E_n - E_m)t}$$

correlation function, defect in gauge theories

Brockmann et al, J. Stat. Mech., 2014
Foda, Zarembo, J. Stat. Mech., 2016
Buhl-Mortensen, de Leeuw, Kristjansen, Zarembo, JHEP, 2016

# Quantum XXZ spin chain

$$H = \sum_{j=1}^{L} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right) \quad \Rightarrow$$

gapped $|\Delta| > 1$

isotropic $|\Delta| = 1$

gapless $0 < |\Delta| < 1$

*additional degeneracies!*

root of unity: $\Delta = \cosh \eta = \frac{q + q^{-1}}{2}$ $\eta = \mathrm{i}\pi \frac{\ell_1}{\ell_2}$ → e.g. $\Delta = \frac{1}{2},\ \eta = \frac{\mathrm{i}\pi}{3}$ → quantum group

$q = \exp \eta$

solution: Bethe ansatz

$$\left[ \frac{\sinh(u_j + \eta/2)}{\sinh(u_j - \eta/2)} \right]^N \prod_{k(\neq j)}^{M} \frac{\sinh(u_j - u_k - \eta)}{\sinh(u_j - u_k + \eta)} = 1$$

Bethe roots ↔ information about eigenstates and eigenvalues of conserved charges

⇕

***spectrum***

*Are Bethe equations enough?*

Bethe, 1931
Hulthén, 1938
Korepin, Bogoliubov Izergin, 1993

# Problem: exact strings

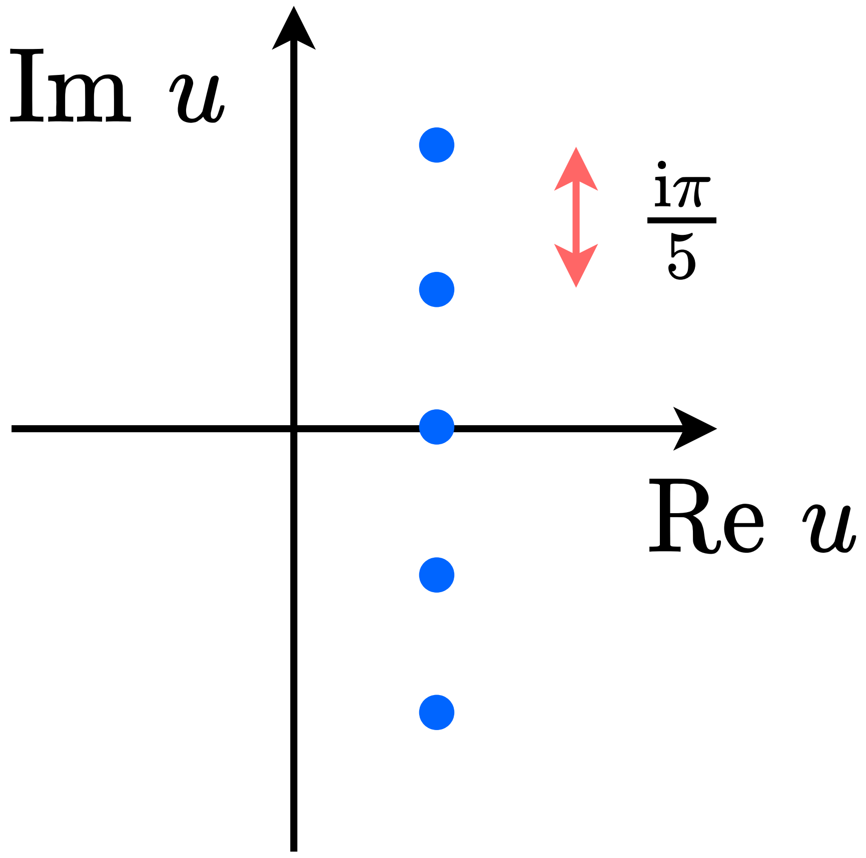
**_exact string (FM)_**

$$\alpha_j = \alpha^{\mathrm{FM}} + \frac{\mathrm{i}\pi}{\ell_2} j$$

$$j = 1, 2, \cdots \ell_2$$

$$\Delta = \cosh\eta, \quad \eta = \mathrm{i}\pi\frac{\ell_1}{\ell_2}$$

e.g. $\eta = \mathrm{i}\pi\frac{2}{5}$




scattering phase $S(u,v) = \dfrac{\sinh(u - v - \eta)}{\sinh(u - v + \eta)}$

as a bound state $\prod_{j=1}^{\ell_2} S(\alpha_j, v) = 1, \quad \forall v$

carries no local conserved charges

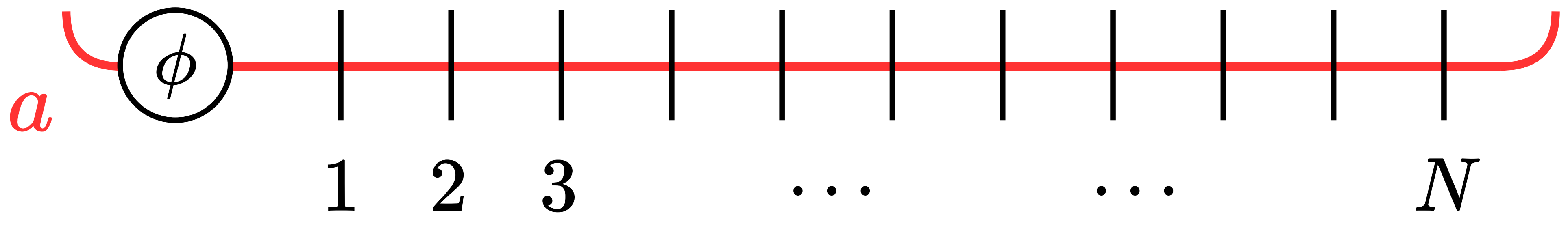
Problem: "trivialise" the Bethe equations?

$$e^{\mathrm{i}p_j N}\prod_{k\neq j}^{M} S(u_j, u_k) = 1 \quad ? \quad e^{\mathrm{i}p_j N}\prod_{k\neq j}^{M} S(u_j, u_k)\prod_{l=1}^{\ell_2} S(u_j, \alpha_l) = 1$$

How to obtain physical solutions with exact (Fabricius—McCoy) strings?

Baxter, Ann. Phys., 1973
Fabricius, McCoy, J. Stat. Phys., 2001
Baxter, J. Stat. Phys., 2002

# 2-parameter transfer matrix




transfer matrix

$$\mathbf{T}_a(u,\phi) = \mathrm{tr}_a\left[\mathbf{L}_{aN}(u)\cdots\mathbf{L}_{a2}(u)\mathbf{L}_{a1}(u)\mathbf{E}_a(\phi)\right]$$

auxiliary space: representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ ⟷ infinite-dimensional complex spin s highest weight rep

***factorisation***

$$\mathbf{L}_{aj}(u,s) = \frac{1}{2}\begin{pmatrix}\mathbf{X}_a^\dagger & 0\\ 0 & 1\end{pmatrix}_j \mathbf{u}_j(x)\begin{pmatrix}\mathbf{W}_a & 0\\ 0 & \mathbf{W}_a^{-1}\end{pmatrix}_j \mathbf{v}_j(y)^{\mathrm{T}}\begin{pmatrix}\mathbf{X}_a & 0\\ 0 & 1\end{pmatrix}_j$$

$\Rightarrow$

$$\mathbf{T}_s^{\mathrm{hw}}(u) = \mathbb{T}(x,y) = \mathbf{Q}(x)\mathbf{P}(y)$$

$$\mathbb{T}(x,y)\mathbb{T}(x',y') = \mathbb{T}(x',y)\mathbb{T}(x,y')$$

$$x = u + \frac{2s+1}{2}\eta, \quad y = u - \frac{2s+1}{2}\eta$$

**Q operator!**

$$\mathbf{L}_{aj}(u) = \sinh(u)\,\frac{\mathbf{K}_a+\mathbf{K}_a^{-1}}{2} + \cosh(u)\,\frac{\mathbf{K}_a-\mathbf{K}_a^{-1}}{2}\,\sigma_j^z + \sinh(\eta)\big(\mathbf{S}_a^+\sigma_j^- + \mathbf{S}_a^-\sigma_j^+\big)$$

$$\mathbf{K}_a\,\mathbf{S}_a^\pm\,\mathbf{K}_a^{-1} = q^{\pm 1}\,\mathbf{S}_a^\pm, \quad \left[\mathbf{S}_a^+,\mathbf{S}_a^-\right] = \frac{\mathbf{K}_a^2-\mathbf{K}_a^{-2}}{q-q^{-1}}$$

*q-deformed*


Lazarescu, Pasquier, J. Stat. Mech., 2014
Vernier, O'Brien, Fendley, J. Stat. Mech., 2019
YM, Caux, Lamers, Pasquier, arXiv: 2011.xxxxx

# Decomposition in auxiliary space

$$\begin{pmatrix} \ddots & & & & \vdots \\ & 0 & * & 0 & 0 & 0 \\ & 0 & 0 & * & 0 & 0 \\ & 0 & 0 & 0 & {\color{red}0} & 0 \\ & 0 & 0 & 0 & 0 & * \\ \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \updownarrow 2s+1$$

***decomposition***

$$\mathbf{T}^{\text{hw}}_{s}(u,\phi) = e^{\mathrm{i}(2s+1)\phi}\,\mathbf{T}^{\text{hw}}_{-s-1}(u,\phi) + \mathbf{T}_{s}(u,\phi)$$

(2s+1)-dimensional half integer spin s unitary rep

Bazhanov, Lukyanov, Zamolodchikov,CMP, 1996
Bazhanov, Łukowski, Menegheli, Staudacher, J.Stat. Mech., 2010

***Wronskian relation***

$$\mathbf{T}_s(u,\phi) = \mathbf{Q}\Big(u+\frac{2s+1}{2}\eta,\phi\Big)\,\mathbf{P}\Big(u-\frac{2s+1}{2}\eta,\phi\Big) - e^{(2s+1)\mathrm{i}\phi}\,\mathbf{Q}\Big(u-\frac{2s+1}{2}\eta,\phi\Big)\,\mathbf{P}\Big(u+\frac{2s+1}{2}\eta,\phi\Big)$$

***matrix TQ relation***

$$\mathbf{T}_{1/2}(u,\phi)\,\mathbf{Q}(u,\phi) = T_0(u-\eta/2)\,\mathbf{Q}(u+\eta,\phi) + e^{\mathrm{i}\phi}\,T_0(u+\eta/2)\,\mathbf{Q}(u-\eta,\phi)$$

For generic ***q***, the construction works.
For root of unity, further simplification!

**Q operator!**

*transfer matrix fusion relations for free!*

YM, Caux, Lamers, Pasquier, arXiv: 2011.xxxxx

# Truncation in auxiliary space

infinite-dimensional complex spin s highest weight rep $\quad \overset{\eta = \mathrm{i}\pi \frac{\ell_1}{\ell_2}}{\longleftrightarrow} \quad$ $\ell_2$-dimensional complex spin s highest weight rep

$$\begin{pmatrix} \ddots & & & & & \vdots \\ & 0 & 0 & 0 & 0 & 0 \\ & * & 0 & 0 & 0 & 0 \\ & 0 & \color{red}{0} & 0 & 0 & 0 \\ & 0 & 0 & * & 0 & 0 \\ \cdots & 0 & 0 & 0 & * & 0 \end{pmatrix} \updownarrow \ell_2$$

auxiliary space

$$\mathbf{T}_s^{\mathrm{hw}}(u,\phi) = e^{\mathrm{i}\ell_2\phi}\,\mathbf{T}_{s-\ell_2}^{\mathrm{hw}}(u,\phi) + \tilde{\mathbf{T}}_s(u,\phi)$$

$$\varepsilon = q^{\ell_2} = e^{\mathrm{i}\pi\ell_1} = \pm 1$$

$$\Updownarrow$$

$$\tilde{\mathbf{T}}_s(u) = \tilde{\mathbb{T}}(x,y) = \tilde{\mathbf{Q}}(x)\tilde{\mathbf{P}}(y)$$

**Q operator!**

matrix TQ relation too

$$\tilde{\mathbf{T}}_s(u,\phi) = \left(1 - \varepsilon^N e^{\mathrm{i}\ell_2\phi}\right)\mathbf{T}_s^{\mathrm{hw}}(u,\phi)$$

$\Rightarrow$ ***truncated fusion and Wronskian relations***

Korff, J. Phs. A, 2004
De Luca, Collura, De Nardis, PRB, 2017

diagonalising Q operator: eigenvalues as Q functions analytically, no problem with exact strings (FM strings)!

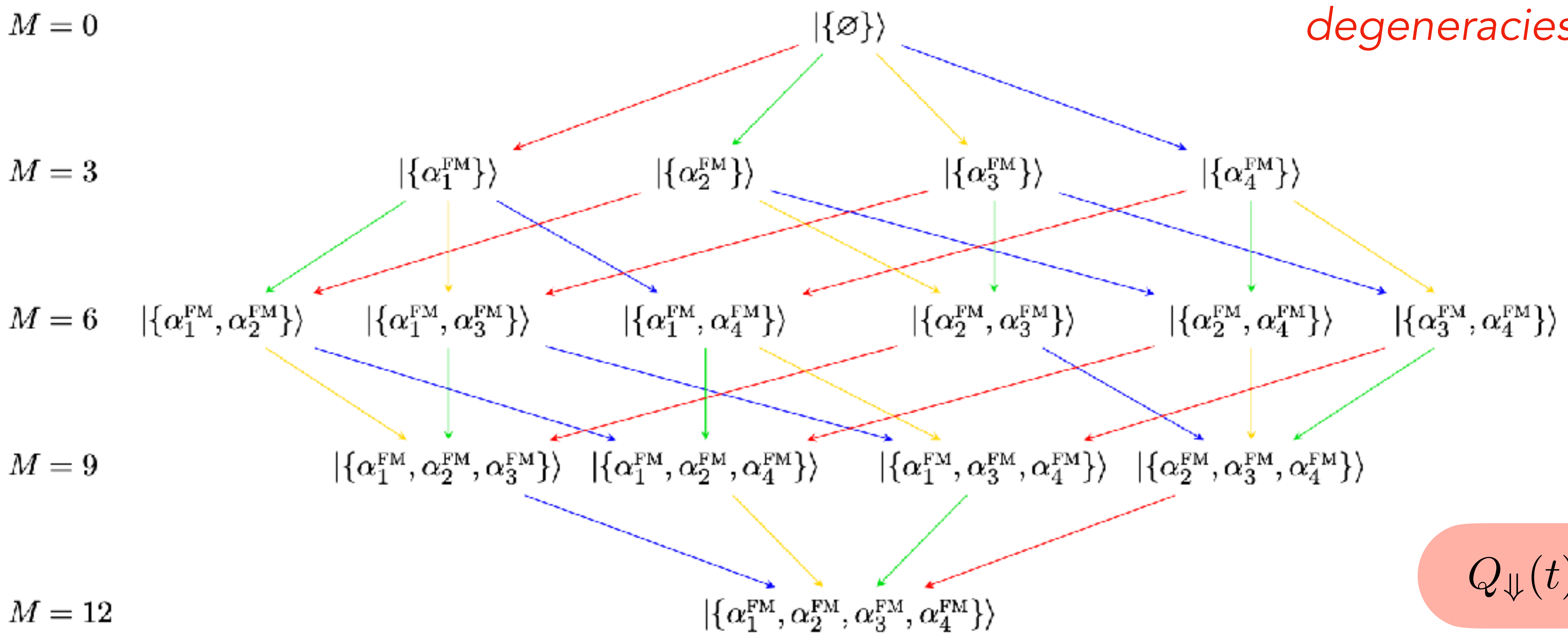
YM, Caux, Lamers, Pasquier, arXiv: 2011.xxxxx

# Descendant tower

Example: $N = 12,\ \Delta = \frac{1}{2},\ \eta = \frac{\mathrm{i}\pi}{3},\ \phi = 0$

*exponential degeneracies*

degenerate in $\mathbf{T}_s(u)$
not in $\tilde{\mathbf{T}}_s(u)$




$$|\{\varnothing\}\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$$

$$|\{\alpha_1^{\rm FM},\alpha_2^{\rm FM},\alpha_3^{\rm FM},\alpha_4^{\rm FM}\}\rangle = |\downarrow\downarrow\cdots\downarrow\rangle$$

$$|\{\alpha_1^{\rm FM}\}\rangle = |\{\alpha_1 - \frac{\mathrm{i}\pi}{3}, \alpha_1, \alpha_1 + \frac{\mathrm{i}\pi}{3}\}\rangle$$

$$Q_{\Downarrow}(t) \propto t^{12} + 220t^{6} + 924 + 220t^{-6} + t^{-12},\ t = e^{u}$$

FM strings within descendant tower:
"free fermion"-like

***proven!***

**analytic results from Q operator**

YM, Caux, Lamers, Pasquier, arXiv: 2011.xxxxx

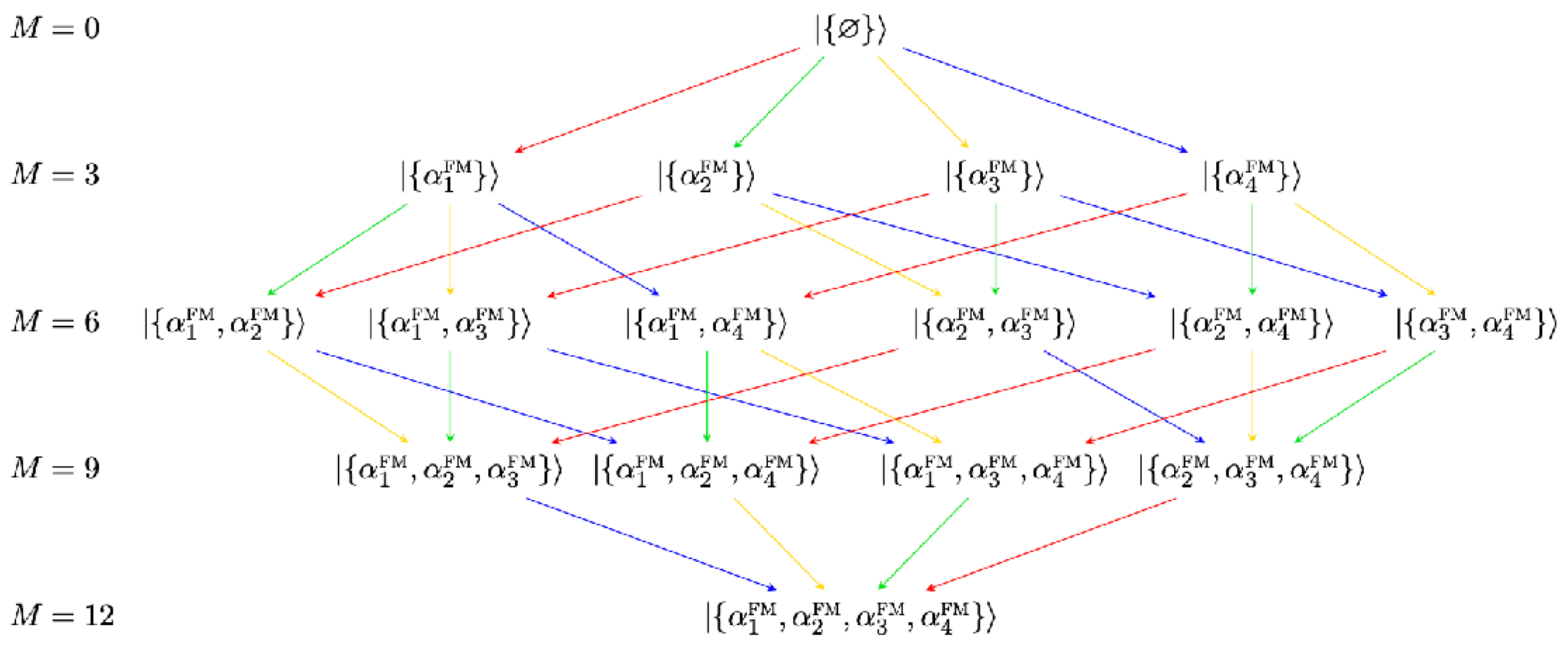
# Conclusion & outlook

Solved the problem with the spectrum of XXZ spin chain at root of unity (descendant tower)

New relations proven (truncated fusion and Wronskian relations)

Spin-1 generalisation (ZF chain)? As a generalisation of Onsager algebra?
(Vernier, O'Brien, Fendley J. Stat. Mech. 2019)

Thermodynamic limit: quasi-local Z charges, non-vanishing spin Drude weight, application to quantum quench

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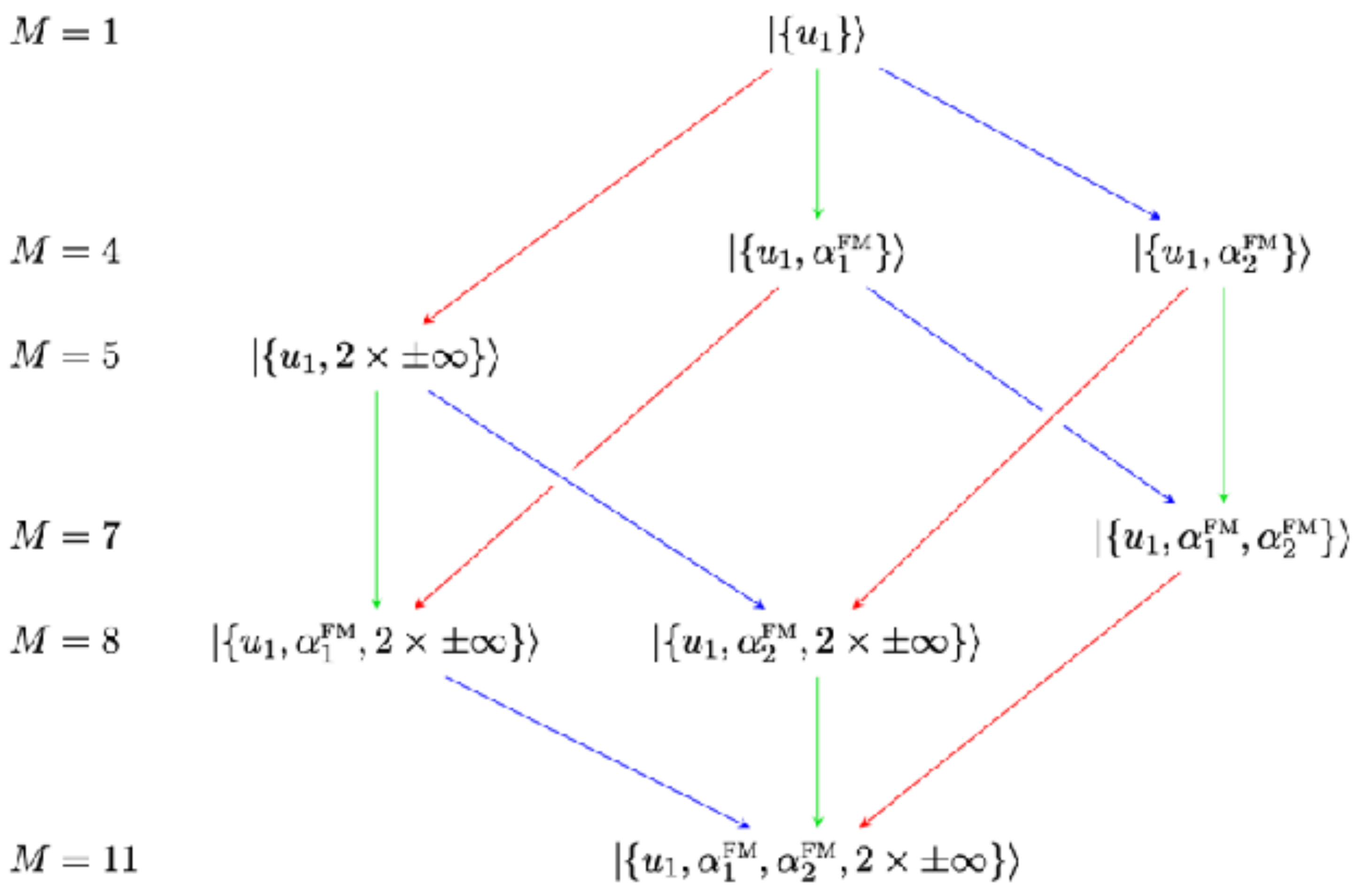

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