CFT in AdS and Gross-Neveu BCFT

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Based on : 2007.04955 (with Simone Giombi) and
2110.04268 (with Simone Giombi, and Elizabeth Helfenberger)

LIJC Nov 25 2021

Plan of the talk

1. Introduction
   - Free scalar example
2. Gross-Neveu BCFT
   - Large N Gross-Neveu model
   - Gross-Neveu-Yukawa (GNY) model in $d = 4 - \epsilon$
3. Conclusions

Introduction

- A BCFT defined on half space preserves $SO(d,1)$ subgroup of the Euclidean conformal group $SO(d+1,1)$. This is also the isometry group of AdS_d, so studying a CFT in AdS is Weyl equivalent to studying a BCFT

$$ds^2 = dz^2 + (d\mathbf{x})^2 = z^2 \left(\frac{dz^2 + (d\mathbf{x})^2}{z^2} \right) = z^2 ds^2_{AdS_d}. \tag{1}$$

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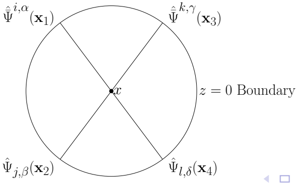
- In the AdS setup, the boundary CFT is located at the boundary of AdS. We can use techniques from the AdS/CFT literature to study properties of BCFT. For instance, the four point function of boundary operators can be computed using Witten diagrams.



A simple example: Free scalar with a boundary

- Consider a free scalar on a half space

$$S = \frac{1}{2}\int_0^\infty dz \int d^{d-1}\mathbf{x}\ (\partial_\mu \phi)^2. \tag{2}$$

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- The usual variational argument gives the equation of motion $\Box\phi = 0$ and the following conformally invariant boundary conditions:
  - Neumann with $\partial_z\phi(z=0,\mathbf{x}) = 0$. In this case, the leading operator living on the boundary is $\hat{\phi}(\mathbf{x}) = \phi(z=0,\mathbf{x})$ which has dimensions $d/2-1$.
  - Dirichlet with $\phi(z=0,\mathbf{x}) = 0$. In this case, the leading operator living on the boundary is $\hat{\phi}(\mathbf{x}) = \partial_z\phi(z=0,\mathbf{x})$ which has dimensions $d/2$.

A simple example: Free scalar with a boundary

- Let's now put the free scalar on AdS_d

$$S = \int d^d x \sqrt{g} \left(\frac{1}{2} (\partial_\mu \phi)^2 - \frac{d(d-2)}{8} \phi^2 \right) . \quad (3)$$

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- Solving the equation of motion gives a bulk-bulk propagator which has the following behaviour near the $z \to 0$ boundary: $\phi(x) \sim z^{\hat{\Delta}} \alpha(\mathbf{x}) + z^{d-1-\hat{\Delta}} \beta(\mathbf{x})$ where

$$\hat{\Delta}(\hat{\Delta} - (d-1)) = -\frac{d(d-2)}{4} \implies \hat{\Delta}_D = \frac{d}{2}, \quad \hat{\Delta}_N = \frac{d}{2} - 1. \tag{4}$$

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- As expected, there are two choices corresponding to Dirichlet and Neumann boundary conditions in flat space.

A simple example: Free scalar with a boundary (q = 1)

- The correlation functions are also related. By method of images,

$$\langle \phi(x_1)\phi(x_2)\rangle^{\text{flat}}_{N/D} = \frac{1}{\left(\mathbf{x}_{12}^2 + (z_1 - z_2)^2\right)^{\frac{d}{2}-1}} \pm \frac{1}{\left(\mathbf{x}_{12}^2 + (z_1 + z_2)^2\right)^{\frac{d}{2}-1}}. \tag{5}$$

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- This is related by a Weyl transform to the scalar bulk-bulk propagator in AdS_d

$$\langle\phi(x_1)\phi(x_2)\rangle^{\text{flat}}_{N/D} = \frac{1}{(z_1 z_2)^{\frac{d}{2}-1}} G^{bb}_{\hat{\Delta}_{N/D}}(\xi), \quad \xi = \frac{\mathbf{x}_{12}^2 + z_{12}^2}{4 z_1 z_2}$$

$$G^{bb}_{\hat{\Delta}} = \frac{1}{(4\xi)^{\hat{\Delta}}}\, {}_2F_1\left(\hat{\Delta}, \hat{\Delta} - \frac{d}{2} + 1, 2\hat{\Delta} - d + 2, -\frac{1}{\xi}\right). \tag{6}$$

Large N Gross-Neveu model

- Let us start with a free massive fermion in AdS which is described by $S = -\int d^d x \sqrt{g} \bar{\Psi}(\gamma \cdot \nabla + m)\Psi$. Fermion satisfies a boundary condition $\gamma_0 \Psi(z \to 0, \mathbf{x}) = \pm\Psi(z \to 0, \mathbf{x})$ and for $m > 0$, the corresponding boundary fermion has dimensions $\hat{\Delta} = (d-1)/2 \mp m$.

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- Next, let us consider $U(N)$ invariant Gross-Neveu model which may be described in AdS by

$$S = -\int d^d x \sqrt{g_x} \left(\bar{\Psi}_i \gamma \cdot \nabla \Psi^i + \sigma \bar{\Psi}_i \Psi^i\right)$$
$$\implies Z = \int [d\sigma] \exp\left(N \mathrm{tr} \log(\gamma \cdot \nabla + \sigma(x))\right) \tag{7}$$

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- At large N, we can do the path integral over σ by assuming a constant saddle $\sigma(x) = \sigma^*$ which can be found by solving

$$\frac{\partial F}{\partial \sigma^*} = -N \mathrm{tr} \left[\frac{1}{\gamma \cdot \nabla + \sigma^*} \right] = 0. \tag{8}$$

- So at leading order in large N, σ^* acts like a mass for the fermions. Choosing $\sigma^* > 0$, for the boundary condition $\gamma_0 \Psi(z \to 0, \mathbf{x}) = -\Psi(z \to 0, \mathbf{x})$ we find the following unitary saddle between $2 < d < 4$

$$\sigma^* = d/2 - 1 \implies \hat{\Delta} = d - \frac{3}{2}. \tag{9}$$

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- For the other choice of boundary condition $\gamma_0 \Psi(z \to 0, \mathbf{x}) = \Psi(z \to 0, \mathbf{x})$, we only find a unitary saddle for $3 \leq d \leq 4$

$$\sigma^* = 2 - d/2 \implies \hat{\Delta} = d - \frac{5}{2}. \tag{10}$$

  Let's call this phase B_2. This is all at leading order in large N.

Sigma fluctuations

- To next order in $1/N$, we need to expand around the saddle to consider σ fluctuations, $\sigma(x) = \sigma^* + \delta\sigma(x)$. The quadratic piece of the effective action is given by

$$S_{\text{eff}}(\sigma) = \frac{N}{2} \int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \, \mathrm{Tr} \left[G_\Psi(x, y) G_\Psi(y, x) \right] \delta\sigma(x) \delta\sigma(y). \tag{11}$$

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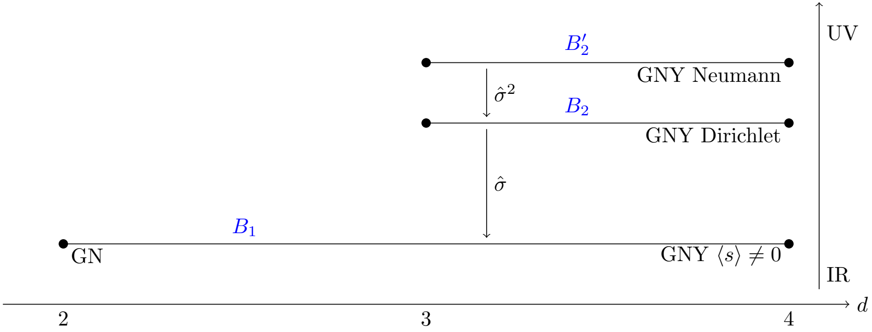
- We need to invert this quadratic piece to obtain σ propagator. For B_1 phase, i.e. $\sigma^* = d/2 - 1$, there is a unique choice of σ propagator, which in particular has a leading boundary scalar of dimension d in its spectrum. However, for $\sigma^* = 2 - d/2$, we have two choices leading to boundary scalars of dimension 2 and $d-3$. We call these phases B_2 and B_2' respectively, and they can only be distinguished at subleading order in $1/N$.

Large N phase diagram



Gross-Neveu-Yukawa (GNY) model in $d = 4 - \epsilon$

- Near four dimensions, we have 3 boundary phases in the large N description. Let's see how they arise in the GNY model which is described by the following action

$$S = \int d^d x \sqrt{g(x)} \Big[\frac{(\partial_\mu s)^2}{2} - \frac{d(d-2)}{8} s^2 - \left(\bar{\Psi}_i \gamma \cdot \nabla \Psi^i + g_1 s \bar{\Psi}_i \Psi^i \right) + \frac{g_2}{24} s^4 \Big]. \tag{12}$$

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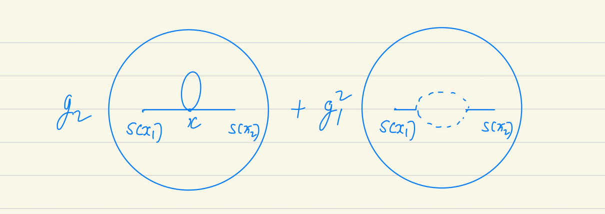
- In the B_1 phase, the scalar s gets a vev determined by the minimum of the potential on hyperbolic space

$$(s^*)^2 = \frac{3d(d-2)}{2g_2^*}. \quad (13)$$

When s has no vev, we can either impose Neumann or Dirichlet boundary conditions on it, which correspond to B_2' and B_2 phase respectively.

Using bulk equations of motion

- I will now demonstrate a simple way to do the perturbation theory in ϵ in this model. Say we want to compute the two-point function of the bulk scalar $\langle s(x_1)s(x_2)\rangle$ to leading order in ϵ. I will demonstrate how it goes in the phases B_2 and B_2' where it does not get any vev. The conventional way is to calculate the following diagrams in AdS



This involves calculating two integrals over AdS.

Using bulk equations of motion

- There is a simpler way to do this calculation if we note that the field s satisfies equations of motion which implies the following equation for the two-point function

$$\left(\nabla^2_{x_2} + \frac{d(d-2)}{4}\right)\left(\nabla^2_{x_1} + \frac{d(d-2)}{4}\right)\langle s(x_1)s(x_2)\rangle = g_1^2\langle\bar{\Psi}_i\Psi^i\rangle^2 + g_1^2\langle\bar{\Psi}_{ia}(x_1)\Psi^{jb}(x_2)\rangle\langle\Psi^{ia}(x_1)\bar{\Psi}_{jb}(x_2)\rangle. \quad (14)$$

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- This is a fourth order differential equation in the cross-ratio ξ for the two-point function $G(\xi)$. We can solve it, and the constants can be fixed by fixing the behaviour at the boundary and by using the bulk CFT data.

Using bulk equations of motion

- Let me write the two-point function explicitly in the B_2 phase, when we impose Dirichlet boundary condition on s

$$G_s^D(\xi) = \frac{1}{4\xi} - \frac{1}{4+4\xi} + \epsilon\Bigg[c_3\left(\frac{\log(\xi)}{1+\xi} - \frac{\log(1+\xi)}{\xi}\right) + \frac{N^2}{2(2N+3)} + \frac{3}{8(2N+3)}\left(\frac{\log 4\xi}{\xi} - \frac{\log(4+4\xi)}{1+\xi}\right)\Bigg]. \tag{15}$$

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- This contains all the BCFT data to order ϵ for the operators appearing in the BOE of s. The dimension of the leading boundary scalar is

$$\hat{\Delta}_s^D = 2 - \frac{\sqrt{4N^2+132N+9} - 2N + 21}{12(2N+3)}\epsilon \tag{16}$$

consistent with the large N result of 2.

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- We also used a similar method to fix fermion two-point function to leading order in ϵ.

Conclusions

- We discussed how hyperbolic space can be used to describe conformal boundaries and applied it to study various boundary phases of Gross-Neveu model. We also used equations of motion to fix the two-point function of bulk fields.
- It will be useful to use this formalism to study other examples of boundary CFT. A natural direction to explore is to gauge the $U(N)$ global symmetry and couple the fermions to the Chern-Simons gauge theory. This may be useful to study how bose-fermi dualities are realized in the presence of a boundary.

Thank You