How coordinate Bethe ansatz works for Inozemtsev model

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NORDITA

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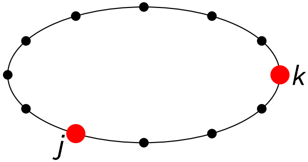
arXiv:2009.14513

with Jules Lamers

Inozemtsev's elliptic spin chain [Inozemtsev, 1989]

$$H \sim \sum_{j<k}^{L} \wp(j-k) \frac{P_{jk}-1}{2}$$



$H : \mathcal{H} \to \mathcal{H}$ with $\mathcal{H} \coloneqq (\mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle)^{\otimes L}$

$$\wp(z) = \frac{1}{z^2} + \sum_{j,k\in\mathbb{Z}}{}^{*} \frac{1}{(z+jL+k\omega)^2} - \frac{1}{(jL+k\omega)^2}$$

$\omega = \mathrm{i}\pi/\kappa$

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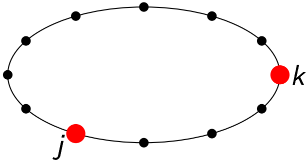

Properties

- $\mathfrak{sl}_2$-invariant, i.e. isotropic
- translation invariant
- pairwise long-range interactions
- periodic

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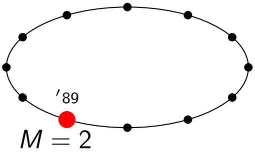

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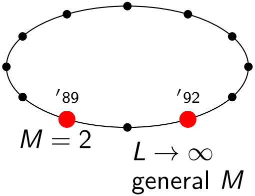

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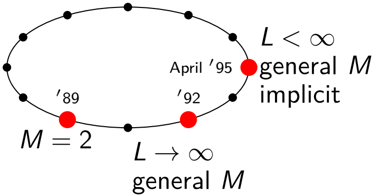

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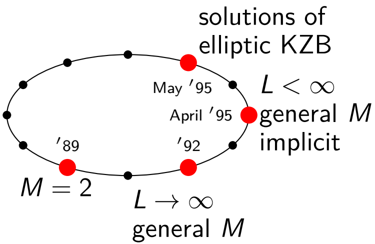

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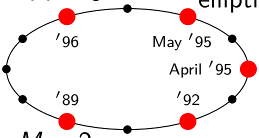

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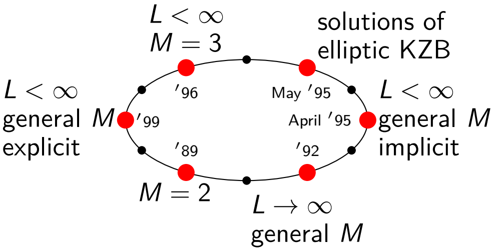

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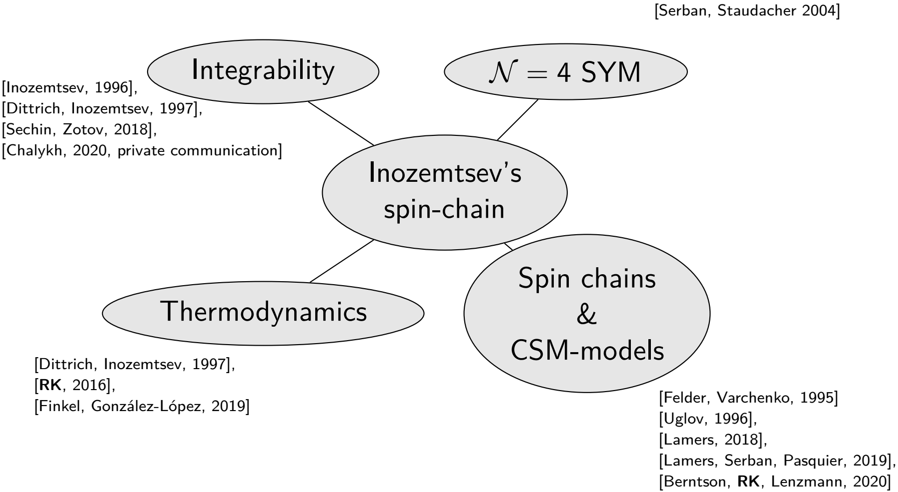
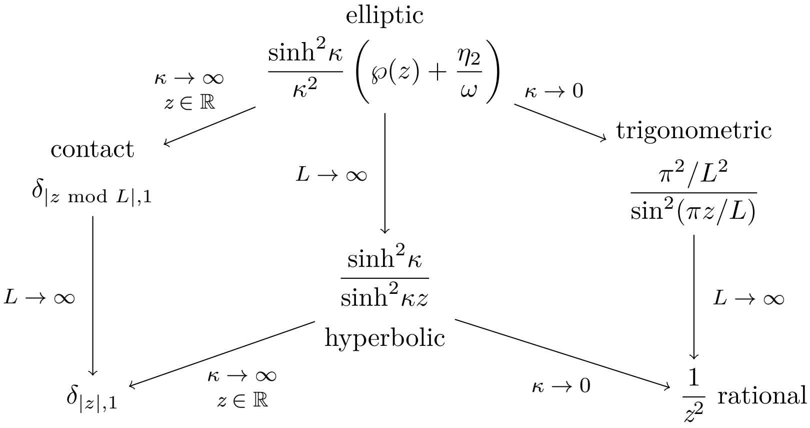
Literature


[Serban, Staudacher 2004]
Integrability
$\mathcal{N} = 4$ SYM
[Inozemtsev, 1996],
[Dittrich, Inozemtsev, 1997],
[Sechin, Zotov, 2018],
[Chalykh, 2020, private communication]
Inozemtsev's spin-chain
Thermodynamics
Spin chains & CSM-models
[Dittrich, Inozemtsev, 1997],
[**RK**, 2016],
[Finkel, González-López, 2019]
[Felder, Varchenko, 1995]
[Uglov, 1996],
[Lamers, 2018],
[Lamers, Serban, Pasquier, 2019],
[Berntson, **RK**, Lenzmann, 2020]

Limits

$$H = -\sum_{j<k}^{L} V(j-k)\,\frac{P_{jk}-1}{2}$$



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Inozemtsev elliptic

$$\frac{\sinh^2\kappa}{\kappa^2}\left(\wp(z)+\frac{\eta_2}{\omega}\right)$$

$\kappa \to \infty$, $z \in \mathbb{R}$ → contact: $\delta_{|z \bmod L|,1}$

$L \to \infty$ → hyperbolic: $\dfrac{\sinh^2\kappa}{\sinh^2\kappa z}$

$\kappa \to 0$ → trigonometric: $\dfrac{\pi^2/L^2}{\sin^2(\pi z/L)}$

contact $\delta_{|z \bmod L|,1}$ — $L \to \infty$ → $\delta_{|z|,1}$

hyperbolic — $\kappa \to \infty$, $z \in \mathbb{R}$ → $\delta_{|z|,1}$

hyperbolic — $\kappa \to 0$ → rational $\dfrac{1}{z^2}$

trigonometric — $L \to \infty$ → rational $\dfrac{1}{z^2}$

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Heisenberg XXX

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Long term goal: Improve our understanding of Inozemtsev's spin chain: is there *integrability* there and if so, what kind?

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New results [**RK**, J. Lamers, 2020]
Our new parametrisation yields

- *Bethe equations*
- all limits for all quantities
- almost *additive* energies
- *rational* equations for the spectral problem
- *extremely good control* over the spectrum

Coordinate Bethe ansatz

Spectral problem

$$H|\Psi\rangle = \varepsilon|\Psi\rangle.$$

Coordinate basis:

$$|{\downarrow\downarrow\uparrow}\dots{\uparrow\downarrow}\dots\uparrow\dots\downarrow\rangle \text{ with } \uparrow \text{ at } \boldsymbol{n} = (n_1, n_2, \dots, n_M)^T$$

Wavefunction component:

$$\langle{\downarrow\uparrow}\dots{\downarrow\uparrow}\dots\uparrow\dots{\downarrow\downarrow}\,|\Psi\rangle = \Psi(\boldsymbol{n})$$

Heisenberg xxx

$$\Psi(\boldsymbol{n}) = \sum_{w\in S_M} A_w(\boldsymbol{p})e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{n}_w}$$

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Quasimomenta: $\boldsymbol{p}$ is quantised by periodicity

Structure of the solution

$$\Psi(\boldsymbol{n}) = \sum_{w \in S_M} \tilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}_w; \varphi) e^{\mathrm{i}(\boldsymbol{p}-\tilde{\boldsymbol{p}})\cdot \boldsymbol{n}_w}$$

Structure of the solution

$$H_{\text{eCSM}}\tilde{\Psi} = \tilde{E}\tilde{\Psi}$$

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$$H_{\text{eCSM}} = -\frac{1}{2}\sum_m \frac{d^2}{dx_m^2} + 2\sum_{m<m'}\left(\wp(x_m - x_{m'}) + \frac{\eta_2}{\omega}\right)$$

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Combining this yields:

Ψ is an eigenfunction with energy $E = \sum_m \varepsilon(p_m) + \tilde{U}$

if

$$\mathrm{CSM\text{-}BAE}(-2\kappa\bar{\rho}_1^{\vee}(\boldsymbol{p}(\varphi)), \varphi) = 0$$

Results: Rationalisation

More precisely

- $\varphi_m = \varphi_m(t_{m,1}, \ldots, t_{m,M})$ ("scattering phases")
- $\mathrm{CSM\text{-}BAE}(-2\kappa\bar{\rho}_1^\vee(\boldsymbol{p}(\varphi)), \boldsymbol{\varphi})$ is an elliptic function in each of the t_α.
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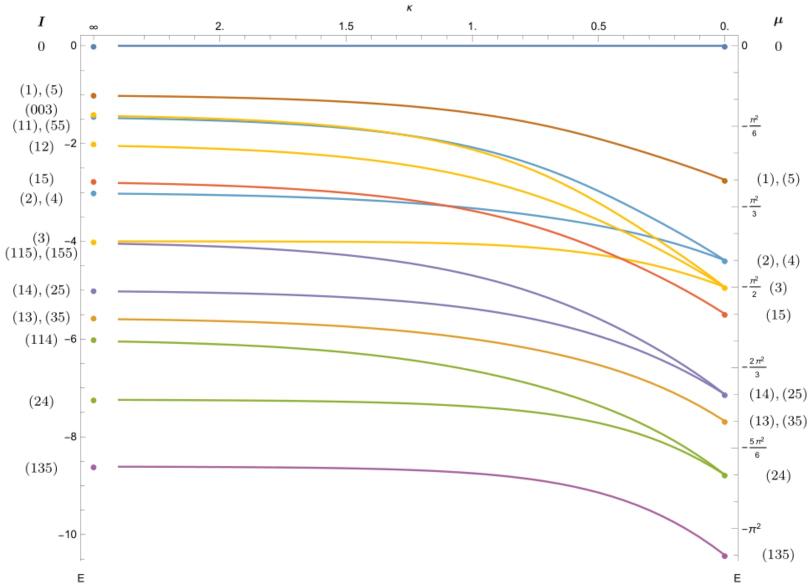
Example: $$\sum_{n=1}^{L-1} \frac{\hat{x}_0 - \hat{x}_n}{\hat{x}_t - \hat{x}_n} = 0,$$

Results: Spectrum at $L = 6$



Results: new observations

State correspondence

Heisenberg	*Haldane-Shastry*
scattering states	Yangian heighest weight states
bound states	affine descendants
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Bound states

Inozemtsev's spectrum has four kinds of bound states that generalise the *critical length equation* of Heisenberg

Take-home message:

- the elliptic spin chain is *easy*
- $E = \sum_m \varepsilon(p_m) + \tilde{U}$, i.e. almost additive energies
- the spectral problem is fully **rational**
- All limits are well-behaved

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- The S-matrix
- How Heisenberg's Bethe integers become Haldane-Shastry motifs
- The complex bound state structure
- Elliptic sums

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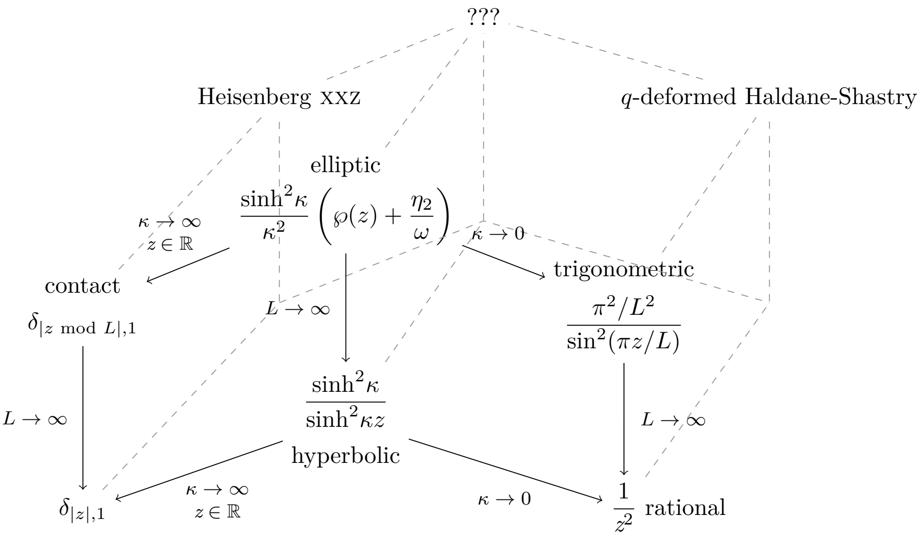
Future directions:

- Connection between two Yangians
- Study completeness for $M > 2$, at least numerically
- XXZ

Limits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q-deformed Haldane-Shastry
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How definitions change everything

Old ingredients:

- $V_{\mathrm{Ino}}(x) = \dfrac{\sinh^2 \kappa}{\kappa^2} \left(\wp(x) + \dfrac{\eta_2}{\omega} \right)$
- $V_{\mathrm{CSM}}(x) = \wp(x)$
- $\tilde{\Psi}_{\tilde{\boldsymbol{p}}} = e^{\mathrm{i}\tilde{\boldsymbol{p}}\cdot x} \sum_{\tau \in S_N} I(\tau) \prod_{\alpha}^{N} \chi_1$
  with multiplicative quasiperiods
- $e^{\mathrm{i}\tilde{p}L}$, $e^{\mathrm{i}\tilde{p}\omega + 2\pi \mathrm{i} q}$, $\quad q = q(\boldsymbol{t})$
- $\rho_1(z) = \zeta(z) - \dfrac{\eta_1}{L} z$
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- U_2 with $F_2(z) = \rho_2' + \rho_2^2 + 3\frac{\eta_2}{\omega}$

Now we use the **Legendre relation** to rewrite everything

When the dust settles

- **Potential**: $V_{\mathrm{Ino}}(x) = \frac{\sinh^2 \kappa}{\kappa^2}\left(\wp(x) + \frac{\eta_2}{\omega}\right)$
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- **Periodicity:** $\boldsymbol{p} = \frac{2\pi}{\omega L}(\boldsymbol{q} + \omega \boldsymbol{l})$
- **Constraints:**

(I) $\bar{\rho}_2\left(\dfrac{\omega p_m}{2\pi}\right) = \tilde{p}_m$

(II) $$\sum_{\beta \in c^{-1}\{c_\alpha - 1,\, c_\alpha + 1\}} \rho_2(t_\alpha - t_\beta) - 2 \sum_{\beta \in (c^{-1}\{c_\alpha\}) \setminus \{\alpha\}} \rho_2(t_\alpha - t_\beta) = \mathrm{i}\left(\tilde{p}_{c(\alpha)} - \tilde{p}_{c(\alpha)+1}\right)$$

$\forall 1 \le \alpha \le N$

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  Sum of scattering phases
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  Sum of scattering phases ($\boldsymbol{q}$) — Bethe counting numbers ($\boldsymbol{I}$)

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When the dust settles

- **Potential**: $V_{\mathrm{SC}}(x) = \wp(x) + \frac{\eta_2}{\omega}$
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  Sum of scattering phases ($\boldsymbol{q}$) — Bethe counting numbers ($\boldsymbol{I}$)
- **Constraints:** $\forall 1 \leq \alpha \leq N$

$$\mathrm{BAE}_\alpha := \sum_{\beta \in c^{-1}\{c_\alpha - 1,\, c_\alpha + 1\}} \rho_2(t_\alpha - t_\beta) - 2 \sum_{\beta \in (c^{-1}\{c_\alpha\}) \setminus \{\alpha\}} \rho_2(t_\alpha - t_\beta) - \mathrm{i}\left(\bar{\rho}_2\left(\frac{q_{c(\alpha)} + \omega I_{c(\alpha)}}{L}\right) - \bar{\rho}_2\left(\frac{q_{c(\alpha)+1} + \omega I_{c(\alpha)+1}}{L}\right)\right) = 0$$

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- **Energy:**

$$\varepsilon_M = \sum_{m=1}^{M} \epsilon(p_m) + \tilde{U}_2^{\text{CSM}}$$