Current mean values in the XYZ model

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Transport in 1D integrable systems

Generalized Hydrodynamics: [Bertini et al., PRL 2016] [Castro-Alvaredo et al., PRX 2016]

- Fluid cells: local GGE $\leftrightarrow \rho_{(n)}(\lambda, x, t)$
- Conserved charges, continuity equations:

$$\partial_t Q_\alpha(x,t) + \partial_x J_\alpha(x,t) = 0 \qquad \alpha \in \mathbb{N}^+$$

- Charge and current mean values:

$$\langle Q_\alpha \rangle = \int d\lambda \rho(\lambda) q_\alpha(\lambda)$$
$$\langle J_\alpha \rangle = \int d\lambda \rho(\lambda) v^{eff}(\lambda) q_\alpha(\lambda)$$

- Flow equation for the root density:

$$\boxed{\partial_t \rho(\lambda) + \partial_x \left(v^{eff}(\lambda)\rho(\lambda)\right) = 0}$$

Setup:

- Integrable system defined on a lattice of size L, with Hamiltonian:

$$H = \sum_{x=1}^{L} h(x)$$

- Set of conserved local charges:

$$[Q_\alpha, Q_\beta] = 0 \qquad\qquad Q_\alpha = \sum_{x=1}^{L} Q_\alpha(x)$$

- Current operators and continuity equation:

$$i\left[H, \sum_{x=x_1}^{x_2} Q_\alpha(x)\right] + (J_\alpha(x_2+1) - J_\alpha(x_1)) = 0$$

- Current mean value?

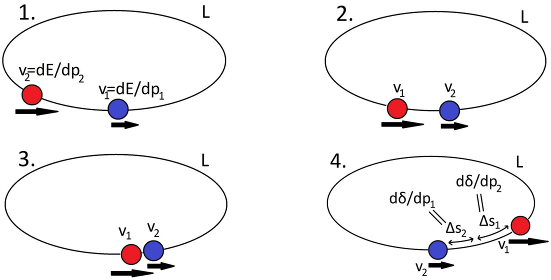
$$\langle\psi|J_\alpha(x)|\psi\rangle = ?$$

XXZ model:

- Form factor expansion **[M. Borsi, B. Pozsgay, LP, PRX 2020]**

$$\langle\psi|J_\alpha(x)|\psi\rangle = \underline{e'}^T \, \underline{\underline{G}}^{-1} \underline{q}_\alpha$$

- Semiclassical picture: $|\psi\rangle \propto \sum_{\mathcal{P}\in S_N} \left(\prod_{j=1}^{N} e^{ix_j p_{\mathcal{P}_j}} \prod_{\substack{j<k \\ \mathcal{P}_j > \mathcal{P}_k}} S(p_j, p_k) \right)$



- What happens if the model does not have $U(1)$ symmetry?

XYZ model

- Hamiltonian:

$$H = \sum_{i=1}^{L} \left[J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right]$$

- R matrix and Yang-Baxter equation:

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & d(\lambda) \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ d(\lambda) & 0 & 0 & a(\lambda) \end{pmatrix} \qquad \begin{aligned} a(\lambda) &= \vartheta_4(\eta)\vartheta_1(\lambda+\eta)\vartheta_4(\lambda) \\ b(\lambda) &= \vartheta_4(\eta)\vartheta_4(\lambda+\eta)\vartheta_1(\lambda) \\ c(\lambda) &= \vartheta_1(\eta)\vartheta_4(\lambda+\eta)\vartheta_4(\lambda) \\ d(\lambda) &= \vartheta_1(\eta)\vartheta_1(\lambda+\eta)\vartheta_1(\lambda) \end{aligned}$$

$$R_{12}(\lambda_{12})R_{13}(\lambda_{13})R_{23}(\lambda_{23}) = R_{23}(\lambda_{23})R_{13}(\lambda_{13})R_{12}(\lambda_{12})$$

- Transfer matrix:

$$t(\lambda) = Tr[T_a(\lambda)] = Tr_a[R_{aL}(\lambda)R_{aL-1}(\lambda)\dots R_{a1}(\lambda)], \qquad [t(\lambda), t(\mu)] = 0$$

$$\frac{d^\alpha}{d\lambda^\alpha} \log\left(t(\lambda)\right)\Big|_{\lambda=0} \propto Q_{\alpha+1}$$

- Algebraic construction of current operators [B. Pozsgay, PRL 2020]

$$\langle\psi|J(\lambda,\mu,x)|\psi\rangle = -\partial_\mu \left(i\frac{d}{d\epsilon}\Lambda^+(\mu|\lambda,\epsilon)\Big|_{\epsilon=0}\right)$$

  Current mean values $\leftrightarrow$ Transfer matrix eigenvalues

- Generalized Algebraic Bethe Ansatz: [L. Takhtadzhan, L. Faddeev, 1979]
  Problem: no reference state $\rightarrow$ Solution: gauge transformation

$$R^l_{an}(\lambda) = M^{-1}_{n+l}(\lambda)R_{an}(\lambda)M_{n+l-1}(\lambda) = \begin{pmatrix}\alpha^l_n(\lambda) & \beta^l_n(\lambda)\\ \gamma^l_n(\lambda) & \delta^l_n(\lambda)\end{pmatrix}$$

  with

$$M_k(\lambda;s,t) = \begin{pmatrix}\vartheta_1(s+k\eta-\lambda) & \frac{1}{g(\tau_k)}\vartheta_1(t+k\eta+\lambda)\\ \vartheta_4(s+k\eta-\lambda) & \frac{1}{g(\tau_k)}\vartheta_4(t+k\eta+\lambda)\end{pmatrix}$$

- Local reference state:

$$\omega^l_n = \vartheta_1(s+(n+l)\eta)e^+ + \vartheta_4(s+(n+l)\eta)e^-$$

$$\gamma^l_n(\lambda)\omega^l_n = 0 \quad \alpha^l_n(\lambda)\omega^l_n = h(\lambda+\eta)\omega^{l-1}_n \quad \delta^l_n(\lambda)\omega^l_n = h(\lambda)\omega^{l+1}_n$$

- Transformation of the monodromy matrix:

$$T_a^l(\lambda) = M_{L+l}^{-1}(\lambda) T_a(\lambda) M_l(\lambda) = \begin{pmatrix} A_l(\lambda) & B_l(\lambda) \\ C_l(\lambda) & D_l(\lambda) \end{pmatrix}$$

- Global reference state:

$$\Omega_l = \omega_1^l \otimes \omega_2^l \cdots \otimes \omega_L^l$$
$$A_l(\lambda)\Omega_l = h^L(\lambda+\eta)\Omega_{l-1} \quad D_l(\lambda)\Omega_l = h^L(\lambda)\Omega_{l+1} \quad C_l(\lambda)\Omega_l = 0$$

- General gauge transformation:

$$T_a^{k,l}(\lambda) = M_k^{-1}(\lambda) T_a(\lambda) M_l(\lambda) = \begin{pmatrix} A_{k,l}(\lambda) & B_{k,l}(\lambda) \\ C_{k,l}(\lambda) & D_{k,l}(\lambda) \end{pmatrix}$$

  Obviously

$$t(\lambda) = A(\lambda) + D(\lambda) = A_{k,k}(\lambda) + D_{k,k}(\lambda)$$

- Commutation relations of the elements of the monodromy matrix can be worked out.

- Eigenstates:

$$\Psi(\lambda_1, \ldots, \lambda_n) = \sum_k B_{k+1,k-1}(\lambda_1) \ldots B_{k+n,k-n}(\lambda_n)\Omega_{k-n}$$

if $n = \frac{L}{2}$ and the rapidities satisfy the Bethe equations:

$$\frac{h^L(\lambda_j + \eta)}{h^L(\lambda_j)} = \prod_{k \neq j} \frac{\alpha(\lambda_k, \lambda_j)}{\alpha(\lambda_j, \lambda_k)}$$

- Eigenvalues:

$$\Lambda(\lambda|\lambda_1, \ldots, \lambda_n) = h^L(\lambda + \eta) \prod_k \alpha(\lambda, \lambda_k) + h^L(\lambda) \prod_k \alpha(\lambda_k, \lambda)$$

- Remaining task: modifying this method for the case of the enlarged transfer matrix