Integrable $\mathcal{E}$-models, 4d Chern–Simons theory & affine Gaudin models.

based on [2008.01829] w/ M. Benini & A. Schenkel
& [2011.13809] w/ S. Lacroix

2d Integrable field theories.

A field theory in 2d is classically integrable if its equations of motion take the Lax form:

$$\text{e.o.m.}(\{\phi_i\}) = 0 \iff \boxed{d\mathcal{L}(z) + \tfrac{1}{2}[\mathcal{L}(z), \mathcal{L}(z)] = 0}$$

Lax connection $\quad \mathcal{L}(z) = \mathscr{L}(\sigma,\tau,z)\, d\sigma + \mathscr{M}(\sigma,\tau,z)\, d\tau$

meromorphic dependence on z.

Example:

Principal chiral model $\quad \partial_+(\underbrace{g^{-1}\partial_- g}_{j_-}) - \partial_-(\underbrace{g^{-1}\partial_+ g}_{j_+}) = 0.$

Field $g \in C^\infty(\Sigma, G)$. Let $j := g^{-1}dg \in \Omega^1(\Sigma, \mathfrak{g})$,

$$\mathcal{L}(z) = \frac{j - z * j}{1 - z^2} = \frac{j_+ \, d\sigma^+}{1 - z} + \frac{j_- \, d\sigma^-}{1 + z}$$

$$\hookrightarrow \; d\mathcal{L}(z) + \tfrac{1}{2}[\mathcal{L}(z), \mathcal{L}(z)] = \frac{1}{1 - z^2}\left(dj + \tfrac{1}{2}[j, j]\right) - \frac{z}{1 - z^2} d * j.$$

Q: What is the origin of the Lax connection?

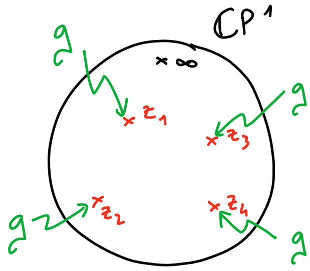
Algebraic/Hamiltonian origin:

* Gaudin models:

$\mathfrak{g}$ semisimple Lie algebra / $\mathbb{C}$, dual bases $\{I_a\}$, $\{I^a\}$.

Lax matrix

$$L(z) := \sum_{i=1}^{N} \frac{I_a \otimes I^{a(i)}}{z - z_i}$$



satisfies

$$\{L_1(z), L_2(w)\} = \left[\frac{I_a \otimes I^a}{w - z}, L_1(z) + L_2(w) \right]$$

Gaudin Hamiltonians:

$$H_i := \mathrm{res}_{z_i} \langle L(z), L(z) \rangle = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{I_a^{(i)} I^{a(j)}}{z_i - z_j}$$

Many finite dimensional integrable systems are representations of Gaudin models.

Example: Neumann model: $\mathfrak{g} = \mathfrak{sl}_2 = \langle E, H, F \rangle$,

$$\sum_{i=1}^{N} \frac{I_a \otimes I^{a(i)}}{z - z_i} + I_a \otimes I^{a(\infty)} \xrightarrow{\text{rep.}} \frac{1}{2} \begin{pmatrix} \sum_{i=1}^{N} \frac{x_i p_i}{z - z_i} & -\sum_{i=1}^{N} \frac{x_i^2}{z - z_i} \\ 1 + \sum_{i=1}^{N} \frac{p_i^2}{z - z_i} & -\sum_{i=1}^{N} \frac{x_i p_i}{z - z_i} \end{pmatrix}$$

$E^{(i)} \longrightarrow \frac{1}{2} p_i^2$, $H^{(i)} \longrightarrow x_i p_i$, $F^{(i)} \longrightarrow -\frac{1}{2} x_i^2$ $\qquad \{x_i, p_j\} = \delta_{ij}$

$E^{(\infty)} \longrightarrow \frac{1}{2}$, $H^{(\infty)} \longrightarrow 0$, $F^{(\infty)} \longrightarrow 0$

* Affine Gaudin models:

$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k \oplus \mathbb{C}d$ untwisted affine KM algebra.

dual bases $\{I^{\tilde{a}}\} = \{I^a \otimes t^n, k, d\}$, $\{I_{\tilde{a}}\} = \{I_a \otimes t^{-n}, d, k\}$

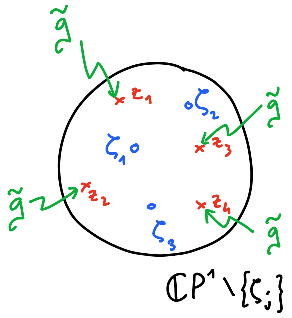
Idea: 2d IFTs with $\mathfrak{g}$-valued Lax connections as representations of Gaudin models associated with $\tilde{\mathfrak{g}}$:

[Feigin-Frenkel '07] [BV '17]

$$L(z)\,dz = \sum_{i=1}^{N} \frac{\sum_{\tilde a} I_{\tilde a}\otimes I^{\tilde a(i)}}{z-z_i}\, dz \xrightarrow{\text{rep.}} \omega\left(\partial_\sigma + \mathcal{L}(\sigma,z)\right)$$

$$\sum_{\tilde a} I_{\tilde a}\otimes I^{\tilde a(i)} = k\otimes d^{(i)} + d\otimes k^{(i)} + \sum_{n\in\mathbb{Z}} I_{a,-n}\otimes I_n^{a(i)}$$

$$\longrightarrow k_i\,\partial_\sigma + I_a \sum_{n\in\mathbb{Z}} e^{-in\sigma} I_n^{a(i)}$$



where $$\boxed{\omega = \sum_{i=1}^{N} \frac{k_i}{z-z_i}\,dz} = \frac{\prod_j (z-\zeta_j)}{\prod_i (z-z_i)}\,dz$$

$$\mathcal{L}(\sigma,z) = \sum_j \frac{\mathcal{F}_j(\sigma)}{z-\zeta_j}$$

$$\{\mathcal{L}_{\underline 1}(\sigma,z), \mathcal{L}_{\underline 2}(\sigma',w)\} = [r_{\underline{12}}(z,w), \mathcal{L}_{\underline 1}(\sigma,z)+\mathcal{L}_{\underline 2}(\sigma',w)]\,\delta(\sigma-\sigma')$$
$$+[s_{\underline{12}}(z,w), \mathcal{L}_{\underline 1}(\sigma,z)-\mathcal{L}_{\underline 2}(\sigma',w)]\,\delta(\sigma-\sigma')$$
$$-2\,s_{\underline{12}}(z,w)\,\partial_\sigma\delta(\sigma-\sigma')$$

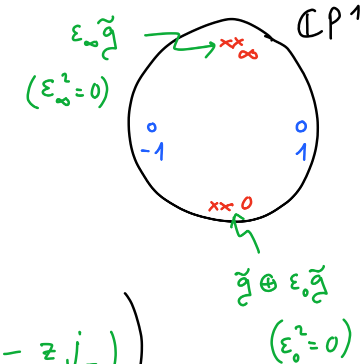
$$\frac{I_a\otimes I^a}{w-z}\,\varphi(w)^{-1} = r+s$$

$$\omega = \varphi(z)\,dz$$

Example: Principal chiral model

$$\left(\overbrace{\frac{I_{\tilde a}\otimes I^{\tilde a(0)}_{[0]}}{z^1}}^{-j_\tau} + \overbrace{\frac{I_{\tilde a}\otimes I^{\tilde a(0)}_{[1]}}{z^2}}^{\partial_\sigma + j_\sigma} - \overbrace{I_{\tilde a}\otimes I^{\tilde a(\infty)}_{[1]}}^{\partial_\sigma}\right)dz$$

$$\xrightarrow{\text{rep.}} \underbrace{\left(\frac{1}{z^2}-1\right)dz}_{\omega_{PCM}}\left(\partial_\sigma + \underbrace{\frac{1}{1-z^2}\left(j_\sigma - z\,j_\tau\right)}_{\mathcal{L}_{PCM}(\sigma,\tau)}\right)$$

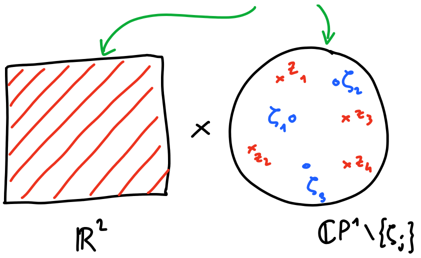


Geometric/Lagrangian origin:

Start from 4d Chern–Simons action:

$$S_\omega(A) = \frac{i}{4\pi}\int_{\Sigma\times C =: X} \omega\wedge CS(A)$$

[Costello '13]
[Costello-Witten-Yamazaki '17, '18]
[Costello-Yamazaki '19]



$$\begin{cases} \bullet\ \omega = \displaystyle\sum_{i=1}^{N} \frac{k_i}{z - z_i}\, dz \\ \bullet\ A = A_\sigma d\sigma + A_\tau d\tau + A_{\bar z} d\bar z \in \Omega^1(X, \mathfrak{g}) \\ \bullet\ CS(A) = \left\langle A, dA + \tfrac{1}{3}[A,A] \right\rangle \end{cases}$$

Remark: ω is singular on surface defects:

$(\underline{z} = \{\text{poles of } \omega\})$ $\qquad D = \bigsqcup_{x \in \underline{z}} \Sigma_x, \qquad \Sigma_x := \Sigma \times \{x\}.$

Nevertheless, $\omega \wedge CS(A)$ is locally integrable near D.

Behaviour of $S_\omega(A)$ under (finite) gauge transformations

$$A \xmapsto{g \in C^\infty(X,G)} {}^gA := -dg\, g^{-1} + g A g^{-1} \quad ?$$

$$S_\omega({}^gA) = S_\omega(A) + \frac{i}{4\pi} \int_X \omega \wedge d\left\langle g^{-1}dg, A \right\rangle$$
$$+ \frac{i}{24\pi} \int_X \omega \wedge \left\langle g^{-1}dg, [g^{-1}dg, g^{-1}dg] \right\rangle$$

Remark: For usual 3d CS, $\quad S_{3d}(A) = \dfrac{k}{4\pi} \displaystyle\int_{M_3} CS(A) \qquad (k \in \mathbb{Z})$

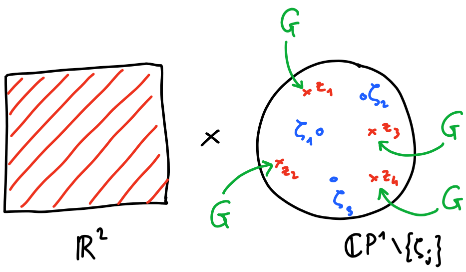
$$S_{3d}({}^gA) = S_{3d}(A) + 2\pi k N, \quad \text{for some } N \in \mathbb{Z}$$

Define defect (Lie) group

$$G^{\underline{z}} := \prod_{x \in \underline{z}} G$$

and defect (Lie) algebra

$$\mathfrak{g}^{\underline{z}} := \prod_{x \in \underline{z}} \mathfrak{g}$$



with bilinear form $\langle\!\langle \cdot , \cdot \rangle\!\rangle_\omega : \mathfrak{g}^{\mathfrak{z}} \times \mathfrak{g}^{\mathfrak{z}} \longrightarrow \mathbb{C}$:

$$\langle\!\langle X, Y \rangle\!\rangle_\omega = \sum_{x \in \mathfrak{z}} k_x \langle X_x , Y_x \rangle \qquad X = (X_x), Y = (Y_x) \in G^{\mathfrak{z}}.$$

Consider embedding

$$\iota : D = \bigsqcup_{x \in \mathfrak{z}} \Sigma_x \hookrightarrow X.$$

<u>Note:</u> Pull back of $g \in C^\infty(X, G)$ & $A \in \Omega^1(X, \mathfrak{g})$ are:

- $\iota^* g \in C^\infty(D, G) = C^\infty\left(\bigsqcup_{x \in \mathfrak{z}} \Sigma_x , G\right)$
$$\cong \prod_{x \in \mathfrak{z}} C^\infty(\Sigma_x, G) \cong C^\infty(\Sigma, G^{\mathfrak{z}})$$
- $\iota^* A \in \Omega^1(D, \mathfrak{g}) \cong \Omega^1(\Sigma, \mathfrak{g}^{\mathfrak{z}})$.

$$S_\omega({}^g A) = S_\omega(A) - \frac{1}{2} \int_\Sigma \langle\!\langle (\iota^* g)^{-1} d_\Sigma (\iota^* g), \iota^* A \rangle\!\rangle_\omega$$
$$+ \frac{1}{12} \int_{\Sigma \times [0,1]} \langle\!\langle \hat{g}^{-1} d\hat{g}, [\hat{g}^{-1} d\hat{g}, \hat{g}^{-1} d\hat{g}] \rangle\!\rangle_\omega$$

[Localised on or "near" defect D.]

$$\hat{g} \in C^\infty(\Sigma \times [0,1], G^{\mathfrak{z}})$$
$$\text{s.t. } \hat{g}|_0 = \iota^* g, \quad \hat{g}|_1 = e.$$

Two ways of ensuring gauge invariance:

Let $k \subset \mathfrak{g}^{\mathfrak{z}}$ be an isotropic Lie subalgebra.

$$\langle\!\langle X, Y \rangle\!\rangle_\omega = 0 \qquad \forall X, Y \in k.$$

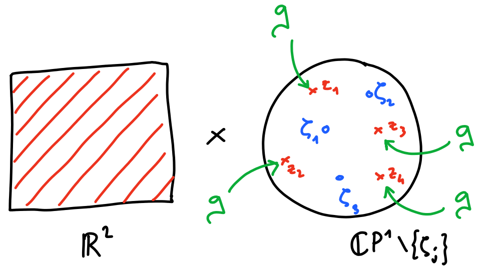
Let $K \subset G^{\mathfrak{z}}$ corresponding connected Lie subgroup.

1) Strict boundary conditions

Consider bulk fields $A \in \Omega^1(X, \mathfrak{g})$ and gauge transformations $g \in C^\infty(X, G)$ satisfying

$\cup$

- $\iota^* A \in \Omega^1(\Sigma, \mathfrak{k}) \subset \Omega^1(\Sigma, \mathfrak{g}^{\underline{z}})$
- $\iota^* g \in C^\infty(\Sigma, K) \subset C^\infty(\Sigma, G^{\underline{z}})$



$\iota^* A = (A|_x)_{x \in \underline{z}} \in \Omega^1(\Sigma, \mathfrak{k})$

$A|_x \in \Omega^1(\Sigma, \mathfrak{g})$

"non-local" boundary condition on $\mathbb{C}P^1$

Theorem (Benini-Schenkel-BV)

$S_\omega(A)$ is gauge invariant.

Summary: Strict boundary conditions imposed via a pullback construction:

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Bulk fields and} \\ \text{gauge transformations} \\ \text{satisfying b.c.}\end{matrix}\right\} \arrow[r, dashed] \arrow[d, dashed] & \underset{\text{Bulk fields}}{\left\{\begin{matrix}\cdot A \in \Omega^1(X, \mathfrak{g}) \\ \cdot g \in C^\infty(X, G)\end{matrix}\right\}} \arrow[d, "\iota^*"] \\
\underset{\text{Defect fields}}{\left\{\begin{matrix}\cdot B \in \Omega^1(\Sigma, \mathfrak{k}) \\ \cdot k \in C^\infty(\Sigma, K)\end{matrix}\right\}} \arrow[r, hook] & \left\{\begin{matrix}\cdot \Omega^1(\Sigma, \mathfrak{g}^{\underline{z}}) \\ \cdot C^\infty(\Sigma, G^{\underline{z}})\end{matrix}\right\}
\end{tikzcd}$$

$\iota^* A = B$
$\iota^* g = k$

2) Homotopical boundary conditions

Gauge fields are not compared by equality but rather via gauge transformations.

$\rightsquigarrow$ Impose boundary conditions via homotopy pullback (i.e. impose them up to gauge transformations):

$$\begin{tikzcd}
\text{homotopy pullback} \arrow[r, dashed] & \underset{}{\overset{\text{Bulk fields}}{\left\{\begin{matrix}\cdot A \in \Omega^1(X, \mathfrak{g}) \\ \cdot g \in C^\infty(X, G)\end{matrix}\right\}}}
\end{tikzcd}$$

$$\left\{\begin{array}{l}\bullet\, B \in \Omega^1(\Sigma, k)\\ \bullet\, k \in C^\infty(\Sigma, K)\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\bullet\, \Omega^1(\Sigma, \mathfrak{g}^{\mathbb{Z}})\\ \bullet\, C^\infty(\Sigma, G^{\mathbb{Z}})\end{array}\right\}$$

Defect fields

A model for the homotopy pullback :

- Fields: $A \in \Omega^1(X, \mathfrak{g})$, $h \in C^\infty(\Sigma, \mathfrak{g}^{\mathbb{Z}})$ $\Big\}$ ${}^{h}(\iota^* A) = B \in \Omega^1(\Sigma, k)$

  edge mode

- Gauge transformations: $(g \in C^\infty(X, G), k \in C^\infty(\Sigma, K))$

$$A \longmapsto {}^{g}A = -dg\, g^{-1} + g A g^{-1}$$
$$h \longmapsto k\, h\, (\iota^* g)^{-1}$$

Couple edge mode to gauge field :

$$S^{ext}_\omega(A, h) := S_\omega(A) - \frac{1}{2}\int_\Sigma \langle\langle h^{-1} d_\Sigma h, \iota^* A\rangle\rangle_\omega$$
$$+ \frac{1}{12}\int_{\Sigma\times[0,1]} \langle\langle \hat{h}^{-1} d\hat{h}, [\hat{h}^{-1} d\hat{h}, \hat{h}^{-1} d\hat{h}]\rangle\rangle_\omega$$

$\hat{h} \in C^\infty(\Sigma \times [0,1])$ s.t. $\hat{h}|_0 = h$, $\hat{h}|_1 = e$.

Theorem (Benini - Schenkel - BV)

Extended action $S^{ext}_\omega(A, h)$ is gauge invariant.

From 4d CS + edge mode to 2d IFTs

Want to turn 4d CS gauge field

$$A = A_\sigma(\sigma, \tau, z)\, d\sigma + A_\tau(\sigma, \tau, z)\, d\tau + A_{\bar{z}}(\sigma, \tau, z)\, d\bar{z}$$

into the 2d IFT Lax connection

$$\mathcal{L}(z) = \mathcal{L}(\sigma,\tau,z)d\sigma + \mathcal{M}(\sigma,\tau,z)d\tau$$

- Step 1: Restrict attention to

$$\mathcal{L} := A = A_\sigma(\sigma,\tau,z)\,d\sigma + A_\tau(\sigma,\tau,z)\,d\tau \in \Omega^{1,0,0}(X,\mathfrak{g})$$

and $g \in C^\infty(X,G)$ such that $\bar\partial g g^{-1} = 0$.

Extended action now reads:

$$S^{ext}_\omega(\mathcal{L},h) = \frac{i}{4\pi}\int_X \omega\wedge\langle\mathcal{L},\bar\partial\mathcal{L}\rangle - \frac{1}{2}\int_\Sigma \langle\!\langle h^{-1}d_\Sigma h, \iota^*\mathcal{L}\rangle\!\rangle_\omega$$

$$\left[{}^h(\iota^*\mathcal{L}) \in \Omega^1(\Sigma,\mathfrak{k})\right] \qquad + \frac{1}{12}\int_{\Sigma\times[0,1]} \langle\!\langle \hat h^{-1}d\hat h, [\hat h^{-1}d\hat h, \hat h^{-1}d\hat h]\rangle\!\rangle_\omega$$

Equations of motion:

* Bulk e.o.m.: $\bar\partial\mathcal{L} = 0$ on $X = \Sigma\times(\mathbb{C}P^1\setminus\{\zeta_j\})$
* Defect e.o.m.: $d_\Sigma(\iota^*\mathcal{L}) + \frac{1}{2}[\iota^*\mathcal{L},\iota^*\mathcal{L}] = 0$ on Σ.

[N.B.: Flatness of $\iota^*\mathcal{L} \in \Omega^1(\Sigma, G^{\frac{z}{}})$ and not of $\mathcal{L} \in \Omega^1(X,G)$.]

- Step 2: Restrict attention to particular solutions of bulk e.o.m.:

Call $\mathcal{L} \in \Omega^{1,0,0}(X,\mathfrak{g})$ admissible if it is:

(a) meromorphic on $\mathbb{C}P^1$ with poles at ζ (zeroes of ω)

(b) such that $\omega\wedge\mathcal{L}$ is bounded near ζ,

(c) and $\omega\wedge\left(d_\Sigma\mathcal{L} + \frac{1}{2}[\mathcal{L},\mathcal{L}]\right)$ is bounded near ζ.

Proposition: If $\mathcal{L} \in \Omega^{1,0,0}(X,\mathfrak{g})$ is admissible then

$$\underbrace{d_\Sigma(\iota^*\mathcal{L}) + \frac{1}{2}[\iota^*\mathcal{L},\iota^*\mathcal{L}] = 0 \text{ on } \Sigma}_{\text{(defect e.o.m.)}} \iff d_\Sigma\mathcal{L} + \frac{1}{2}[\mathcal{L},\mathcal{L}] = 0 \text{ on } X.$$

Extended action now reads:

$$S^{ext}_\omega(\mathcal{L},h) = -\frac{1}{2}\int_\Sigma \langle\!\langle h^{-1} d_\Sigma h, \iota^*\mathcal{L}\rangle\!\rangle_\omega + \frac{1}{12}\int_{\Sigma\times[0,1]} \langle\!\langle \hat h^{-1} d\hat h, [\hat h^{-1} d\hat h, \hat h^{-1} d\hat h]\rangle\!\rangle_\omega$$

$$\left[{}^h(\iota^*\mathcal{L}) \in \Omega^1(\Sigma, k)\right]$$

and equations of motion now read:

$$\boxed{d_\Sigma \mathcal{L} + \frac{1}{2}[\mathcal{L},\mathcal{L}] = 0} \quad \text{on } X.$$

- Step 3: Solve the constraint

$${}^h(\iota^*\mathcal{L}) \in \Omega^1(\Sigma, k)$$

for admissible $\mathcal{L}$ in terms of $h \in C^\infty(\Sigma, G^{\underline z})$.

We assume ω has a double pole at ∞, and partially fix gauge invariance to fix components of $h \in C^\infty(\Sigma, G^{\underline z})$ at ∞: $\quad (\underline z' := \underline z \setminus \{\infty\})$

$$h \in C^\infty(\Sigma, G^{\underline z}) \rightsquigarrow \ell \in C^\infty(\Sigma, G^{\underline z'})$$

$$\iota : D = \bigsqcup_{x\in \underline z} \Sigma_x \hookrightarrow X \quad \rightsquigarrow \quad j : D' = \bigsqcup_{x\in \underline z'} \Sigma_x \hookrightarrow X$$

$${}^h(\iota^*\mathcal{L}) \in \Omega^1(\Sigma, k) \rightsquigarrow {}^\ell(j^*\mathcal{L}) \in \Omega^1(\Sigma, k').$$

$$k = \varepsilon_\infty \mathfrak{g} \oplus k' \subset \mathfrak{g}^{\underline z}$$

To satisfy admissibility conditions (a) & (b), take:

$$\mathcal{L} = \sum_{y\in\underline\zeta} \frac{\mathcal{L}_y}{z-y}$$

- $\underline\zeta$ set of zeroes of ω
- $\mathcal{L}_y \in \Omega^1(\Sigma, \mathfrak{g})$

To satisfy admissibility condition (c), require

$$\mathcal{E}(j^*\mathcal{L}) = *(j^*\mathcal{L}) \qquad [\check{S}evera\ '16]$$

where $\mathcal{E} : \mathfrak{g}^{\underline z'} \xrightarrow{\cong} \mathfrak{g}^{\underline z'}$ is defined by $(\underline\zeta = \underline\zeta^+ \sqcup \underline\zeta^-)$

$$\begin{tikzcd}
\mathfrak{g}^{\underline{z}'} \ni X \arrow[d, "\mathcal{E}"'] & \displaystyle\sum_{y\in\underline{\xi}^+} \frac{f_y}{z-y} + \sum_{y\in\underline{\xi}^-} \frac{f_y}{z-y} \arrow[l, "\cong\,(|\underline{\xi}|=|\underline{z}'|)"', "j^*"] \arrow[d] \\
\mathfrak{g}^{\underline{z}'} \ni \mathcal{E}X & \displaystyle\sum_{y\in\underline{\xi}^+} \frac{f_y}{z-y} - \sum_{y\in\underline{\xi}^-} \frac{f_y}{z-y} \arrow[l, "\cong"', "j^*"]
\end{tikzcd}$$

We have (under mild assumptions)

$$\mathfrak{g}^{\underline{z}'} = \mathrm{Ad}_\ell^{-1}\mathfrak{k}' \oplus \mathcal{E}\,\mathrm{Ad}_\ell^{-1}\mathfrak{k}' \qquad \forall \ell \in G^{\underline{z}'}.$$

Define $P_\ell : \mathfrak{g}^{\underline{z}'} \longrightarrow \mathcal{E}\,\mathrm{Ad}_\ell^{-1}\mathfrak{k}'$ projector with $\ker P_\ell = \mathrm{Ad}_\ell^{-1}\mathfrak{k}'$.

Proposition. (Lacroix - BV) There is a unique solution of the constraint $\ell(j^*\mathcal{L}) \in \Omega^1(\Sigma, \mathfrak{k}')$ such that $\mathcal{E}(j^*\mathcal{L}) = *(j^*\mathcal{L})$, given by

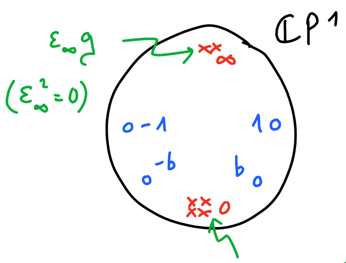
$$j^*\mathcal{L} = (1-P_\ell)\,\mathcal{E}(\ell^{-1} * d_\Sigma \ell) + P_\ell(\ell^{-1}d_\Sigma \ell).$$

Resulting 2d integrable σ-model on $K'\backslash G^{\underline{z}'}$,

$$S^{ext}_\omega(\ell) = -\frac{1}{2}\int_\Sigma \langle\langle \ell^{-1}d_\Sigma\ell, (1-P_\ell)\mathcal{E}(\ell^{-1} * d_\Sigma \ell) + P_\ell(\ell^{-1}d_\Sigma\ell)\rangle\rangle_\omega + \frac{1}{12}\int_{\Sigma\times[0,1]} \langle\langle \hat{\ell}^{-1}d\hat{\ell}, [\hat{\ell}^{-1}d\hat{\ell}, \hat{\ell}^{-1}d\hat{\ell}]\rangle\rangle_\omega$$

known as an $\mathcal{E}$-model. [Klimčík - Ševera '95]

Example: $\omega = \dfrac{(z^2-1)(b^2-z^2)}{z^4}\,dz \qquad (1>b>0)$



$$\mathfrak{g}^{\underline{z}'} := \mathfrak{g} \oplus \varepsilon_0\mathfrak{g} \oplus \underbrace{\varepsilon_0^2\mathfrak{g} \oplus \varepsilon_0^3\mathfrak{g}}_{\mathfrak{k}'} \qquad (\varepsilon_0^4=0)$$

$$S^{ext}_\omega(g,u) = \int_\Sigma \Big(\tfrac{1}{2}(1-b)\langle dg g^{-1}, *dg g^{-1}\rangle + \tfrac{1}{2}b(1-b)\langle du, *du\rangle - b\langle dg g^{-1}, du\rangle - \tfrac{1}{6}b^2\langle u, [du, du]\rangle \Big)$$

$g \in C^\infty(\Sigma, G)$

$u \in C^\infty(\Sigma, \mathfrak{g})$

Conclusions:

- Passage from 4d to 2d via edge modes:

$$\begin{tikzcd} & \text{4d CS w/edge modes} & \\ \text{4d CS w/ b.c.} \arrow[ur] & & \text{2d } \sigma\text{-model } (\mathcal{E}\text{-model}) \arrow[ul] \end{tikzcd}$$

  * Edge modes via homotopy pull back.
    [Mathieu – Murray - Schenkel - Teh '20]
  * Generalises to higher order poles in ω.
  * More general solution to constraint?

- Hamiltonian analysis?
  * Hamiltonian integrability (r-matrix, ...)
  * Affine Gaudin model description? [BV'19]

$$\left\{\begin{matrix}\text{boundary conditions} \\ \text{in 4d CS theory}\end{matrix}\right\} \overset{?}{\longleftrightarrow} \left\{\begin{matrix}\text{representations} \\ \text{of } \tilde{\mathfrak{g}}\end{matrix}\right\}$$

- Quantization
  * Relationship between quantum 4d CS & 2d QIFT ?
  * Quantum 4dCS vs. affine quantum Gaudin model ?
  * ...