Towards

Lefschetz thimbles in field theory

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based on
NN arXiv/1802.04202
I.Krichever, NN arXiv/2010.15575
A.Ilina,I.Krichever,NN arXiv/1903.01778

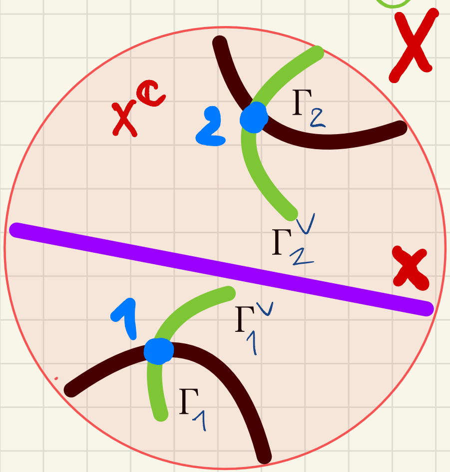
analytic function, multi-valued

$$I_\hbar(t) = \int_X e^{-\frac{S_t(x)}{\hbar}}\, dx = \sum_a n_a \int_{\Gamma_a} e^{-\frac{S}{\hbar}}\, \Omega$$

$\#(X \cap \Gamma_a)^\vee$ dual Lefschetz thimble

partition function of TFT in +1 dimensions

analytic top degree (n,0)-form on $X^{\mathbb{C}}$



Γ_a — middle dimensional contour = relative cycle $\subset X^{\mathbb{C}}$

basis in $H_{\dim(X)}(X^{\mathbb{C}}, X^{\mathbb{C}}_+)$

$\mathrm{Re}\left(\frac{S}{\hbar}\right)^{-1}(\mathbb{R}_{\gg 0})$

$$[X] = \sum_a n_a \Gamma_a$$

Exponential integrals and Lefschetz thimbles

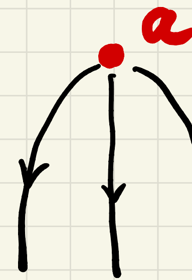
Optimal representative for Γ_a

start at some critical point and run steepest descent

$a \longleftrightarrow \mathrm{Crit}(S)$

$$\dot{x} = \nabla^h \mathrm{Re}(S/\hbar)$$

some hermitian metric



$\mathrm{Re}(S/\hbar)$ grows, hence $\exp(-S/\hbar)$ decreases in abs value

analyticity of $S \implies$ every (isolated) critical point of $\mathrm{Re}(S/\hbar)$ has Morse index = dim(X)

near non degenerate critical point...

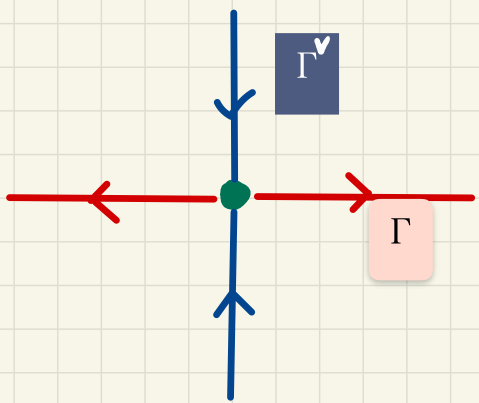
$$S(z) = \sum_{i=1}^{n} z_i^2$$

$$\frac{\vec{z}}{\sqrt{\hbar}} = \vec{x} + i\vec{y}$$

Stokes phenomena
résurgence
Picard-Fuchs
Gauss-Manin
WKB
singularity theory

$$\mathrm{Re}(S/\hbar) = \sum_{i=1}^{n} x_i^2 - y_i^2$$

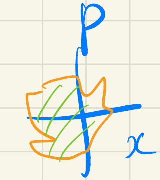
Lefschetz thimble $y = 0, \quad \dot{x} = x$



dual Lefschetz thimble (ħ—>-ħ)

$\dot{y} = -y, \quad x = 0$

Exponential integrals in quantum mechanics



Hermitian operator

$$\mathrm{Tr}_{\mathscr{H}}\left(e^{-\beta \hat{H}}\right)$$

Hilbert space

Wick rotation

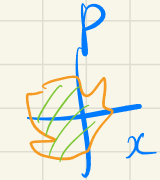
$$= \int_X [D_p D_x]\, e^{\frac{i}{\hbar}\oint p\,dx \,-\, \beta \int_0^1 H(p(t),x(t))\,dt}$$

$X = L\mathcal{P}$ **réal (analytic) symplectic manifold**

$X = \{\ (p(t),x(t))\ \mid\ t\sim t+1\ \}$

$(p(t),x(t)) = \gamma(t)$

Exponential integrals in quantum mechanics



Hermitian operator

$$\mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta \hat{H}}\, g \right)$$

symmetry group element

Hilbert space

Wick rotation

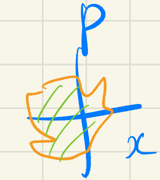
$$= \int_{X_g} [D p\, D x]\; e^{\frac{i}{\hbar}\oint p\,dx \;-\; \beta \int_0^1 H(p(t),x(t))\,dt}$$

$$X_g = L_g \mathcal{P}$$

réal (analytic) symplectic manifold of g-twisted loops

$\{\ (p(t),x(t)) \mid (p(t+1),x(t+1)) = g\,(p(t),x(t))\ \}$

Exponential integrals in quantum mechanics



Hermitian operator

$$\mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta \hat{H}}\, g \right)$$

Hilbert space

symmetry Lie group G element $g = P\exp \oint A_t\, dt$

Wick rotation

$\mu : \mathcal{P} \longrightarrow \mathfrak{g}^*$

$$= \int_X [D_p D_x]\, e^{\frac{i}{\hbar}\oint p\,dx \;-\; \beta \int_0^1 H(p(t),x(t))\,dt \;+\; \frac{i}{\hbar}\oint \langle \mu, A_t \rangle\, dt}$$

Both β and g can be complex now

$X = L\mathcal{P}$

réal (analytic) symplectic manifold of periodic loops

$\{ (p(t),x(t)) \mid (p(t+1),x(t+1)) = (p(t),x(t)) \}$

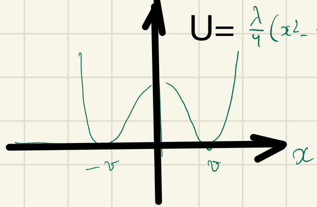
$X^{\mathbb{C}} = L\mathcal{P}^{\mathbb{C}}$

Schrödinger equation

$$\hat{H}\psi = E\psi$$

$$\hat{H} = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2} + U(x)$$

$U = \frac{\lambda}{4}(x^2 - v^2)^2$



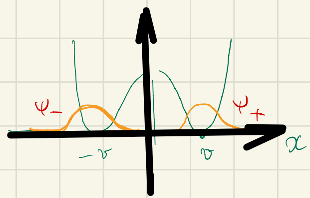
Parity symmetry

$$\hat{P}\,\psi(x) = \psi(-x)$$

Classically, two identical vacua

$x = v$, $p = 0$

$x = -v$, $p = 0$



Quantum mechanically, for small $\hbar$

look for
nearly gaussian,
localized near $x = \pm v$

$$U \approx \lambda v^2 (x \mp v)^2 = \frac{1}{2}\omega_0^2 (x \mp v)^2$$

perturbatively:
doubly degenerate spectrum

$$E = \frac{1}{2}\hbar\omega_0 + \sum_{k\geq 1} \hbar^k \varepsilon_l$$

$$\hat{H}\psi_\pm = E\psi_\pm \qquad \hat{P}\psi_\pm = \psi_\mp$$

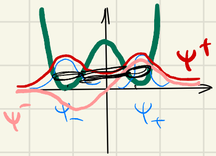
However, degeneracy is not possible, at least not for the ground state

$$E = \min_{\|\psi\|^2 = 1} \int \psi^* \hat{H} \psi$$

Feynman’ s argument

Nonperturbatively

$$\boxed{\hat{H}\,\Psi^{\pm} = E_{\pm}\Psi^{\pm}\,, \qquad \Psi^{\pm} = \frac{1}{\sqrt{2}}\left(\Psi_{+} \pm \Psi_{-}\right), \qquad \hat{P}\,\Psi^{\pm} = \pm\,\Psi^{\pm}\,,}$$



$E_{-} > E_{+}$

$$E_{-} - E_{+} \sim c\,\exp\left(-S_0/\hbar\right)$$ instanton-anti instanton contribution

classic (semi-classical) approach to computing $\Delta E = E_{+} - E_{-}$

estimate the matrix elements

$$\langle \pm v |\, e^{-\beta \hat{H}} \,| \pm v \rangle = e^{-\beta E_{+}} \langle \pm v | \psi^{+} \rangle \langle \psi^{+} | \pm v \rangle + e^{-\beta E_{-}} \langle \pm v | \psi^{-} \rangle \langle \psi^{-} | \pm v \rangle + \dots$$

$$= \exp(-\beta E_{+}) \times \left(1 \pm c_1\, e^{-\beta \Delta E}\right) |\psi^{+}(v)|^2 + \dots$$

$$\beta \longrightarrow \infty$$

$$e^{\frac{i}{\hbar}\oint p\,dx \;-\; \beta\int_0^1 H(p,x)\,dt} \;=\; \mathrm{Tr}\, e^{-\beta\hat{H}}$$

$$= e^{-S/\hbar}$$

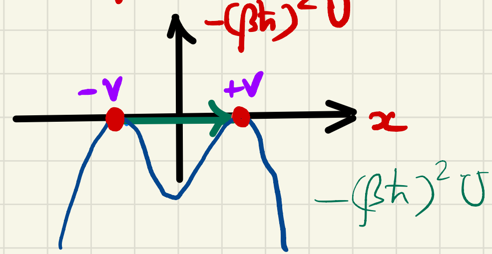
$$\mathrm{Crit}\left(\frac{S}{\hbar}\right)$$

solution of (almost) Hamilton equations

$$\begin{cases} \dfrac{i}{\hbar}\dot{x} = \beta\dfrac{\partial H}{\partial p} = \beta p \\[2ex] -\dfrac{i}{\hbar}\dot{p} = \beta\dfrac{\partial H}{\partial x} = \beta U'_x \end{cases}$$

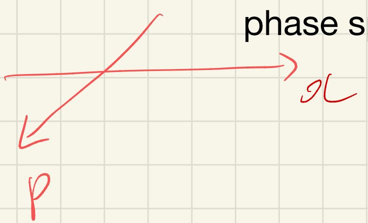
easy to miss
in the coordinate picture

$$\ddot{x} = (\beta\hbar)^2 U'_x$$



inverted potential

the critical orbit
sticks out to the complex
phase space

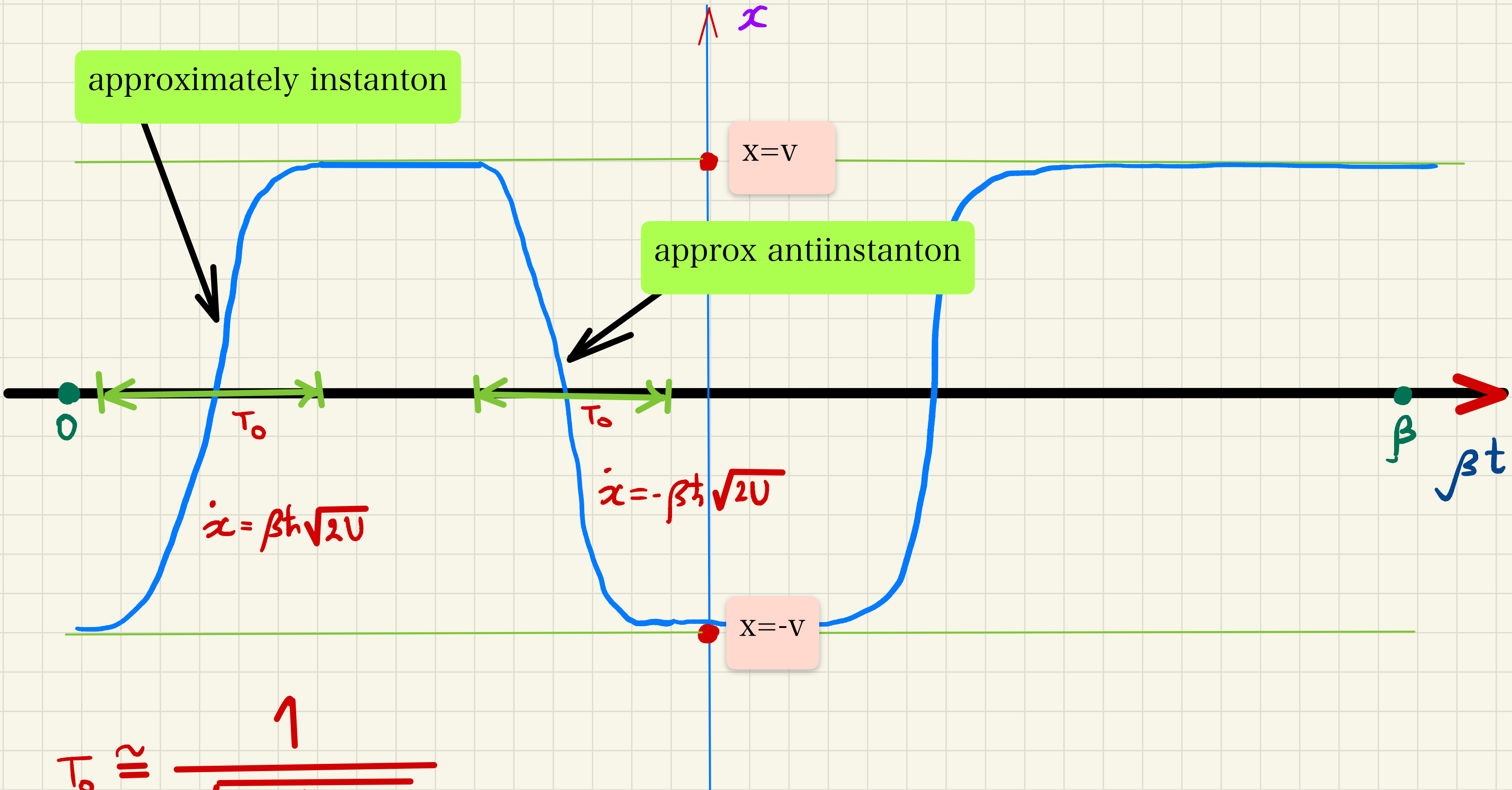


2 boundary conditions:

1) x(0)=-v , x(1)=+v

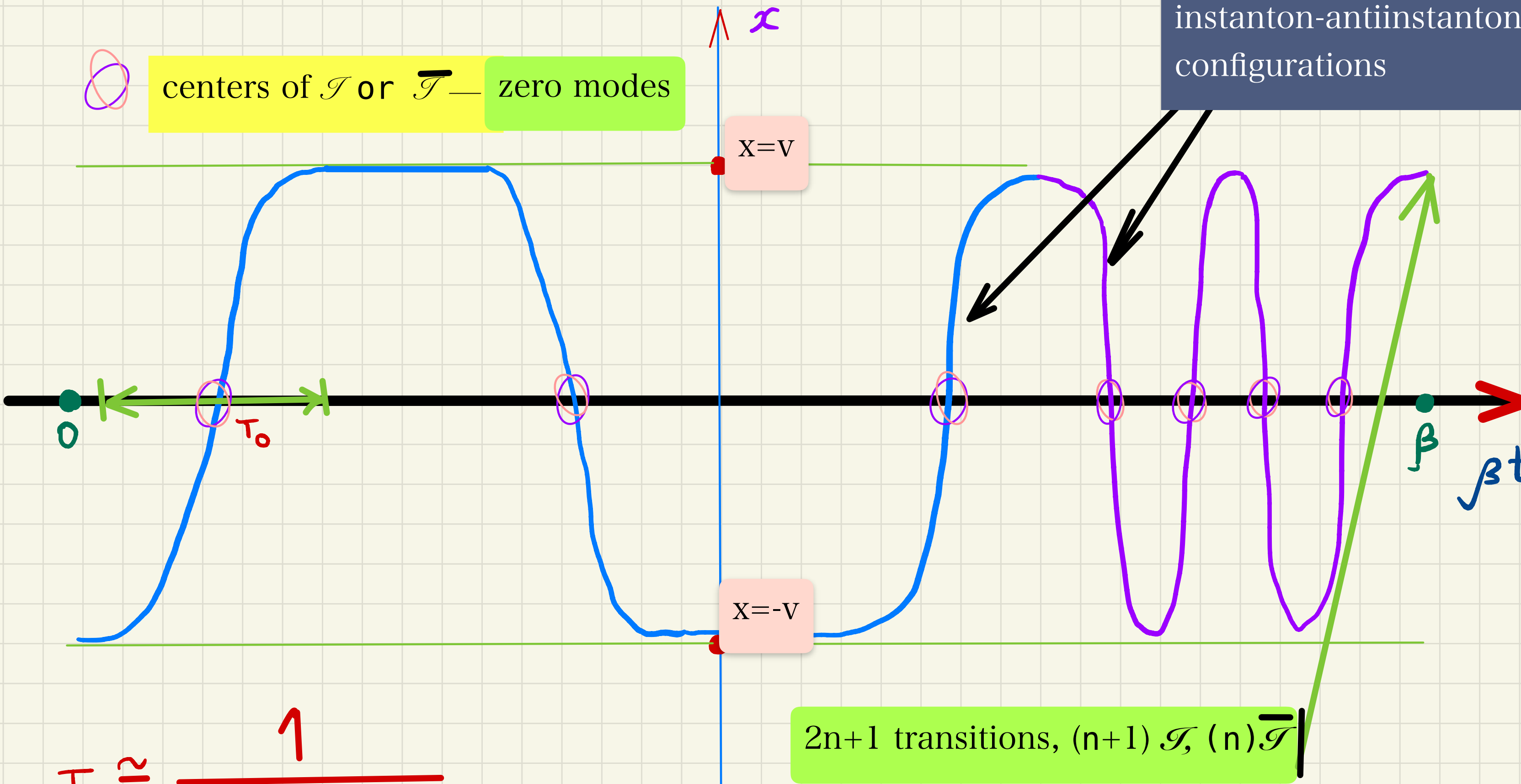
2) x(0)=x(1)=+v

Textbook calculation for β real: case 1)



$$T_0 \cong \frac{1}{\sqrt{U''_{xx}(V)}}$$

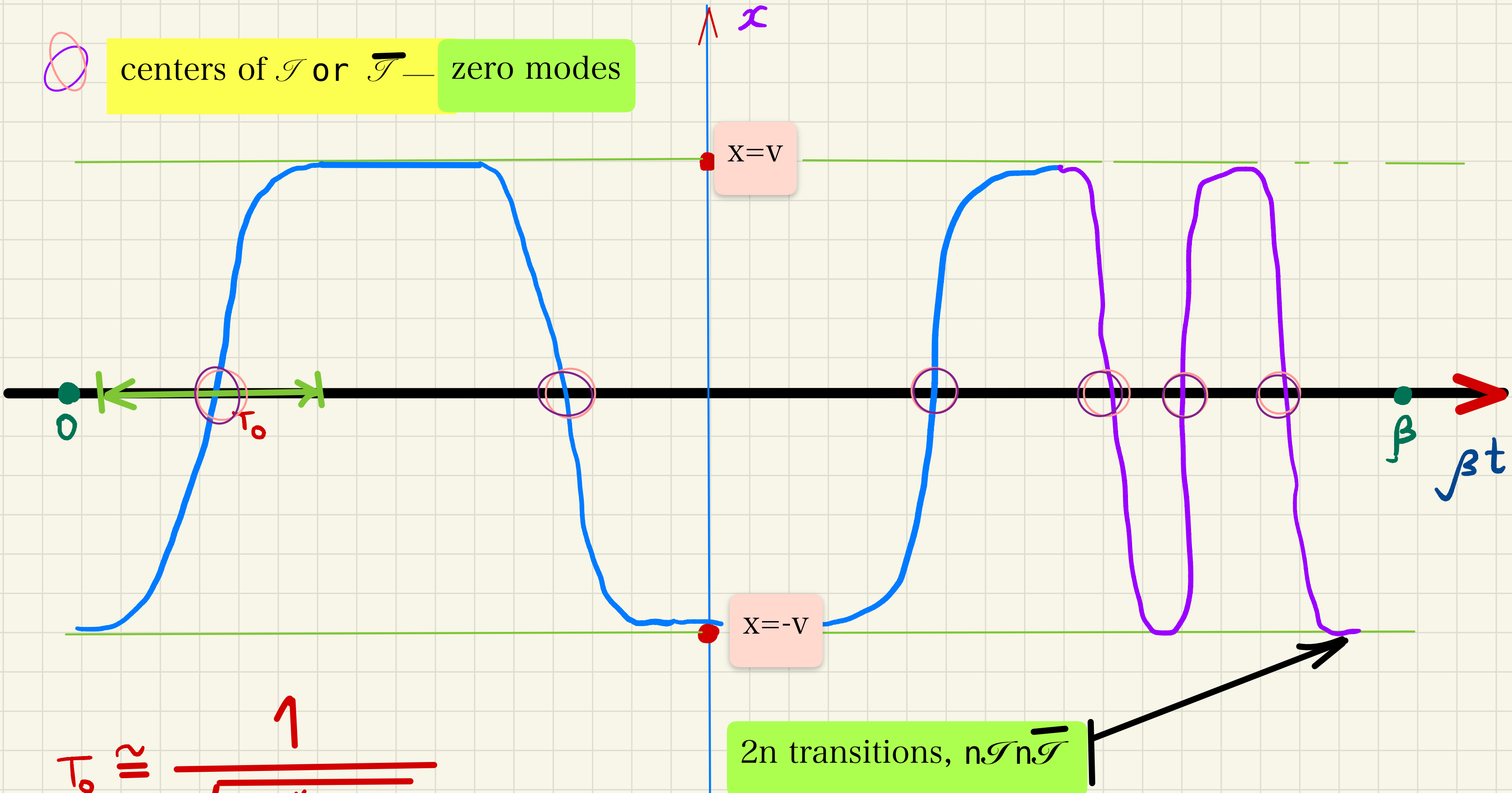
Textbook calculation for $\sqrt{\beta}$ real: case 1)



$$T_0 \cong \frac{1}{\sqrt{U''_{xx}(v)}}$$

Textbook calculation for β real: case 2)

centers of $\mathscr{I}$ or $\bar{\mathscr{I}}$ — zero modes



$$\tau_0 \cong \frac{1}{\sqrt{U''_{xx}(v)}}$$

$$\delta S = \int f_t\, \delta\xi_t + g(t,t')\, \delta\xi_t\, \delta\xi_{t'} + \dots$$

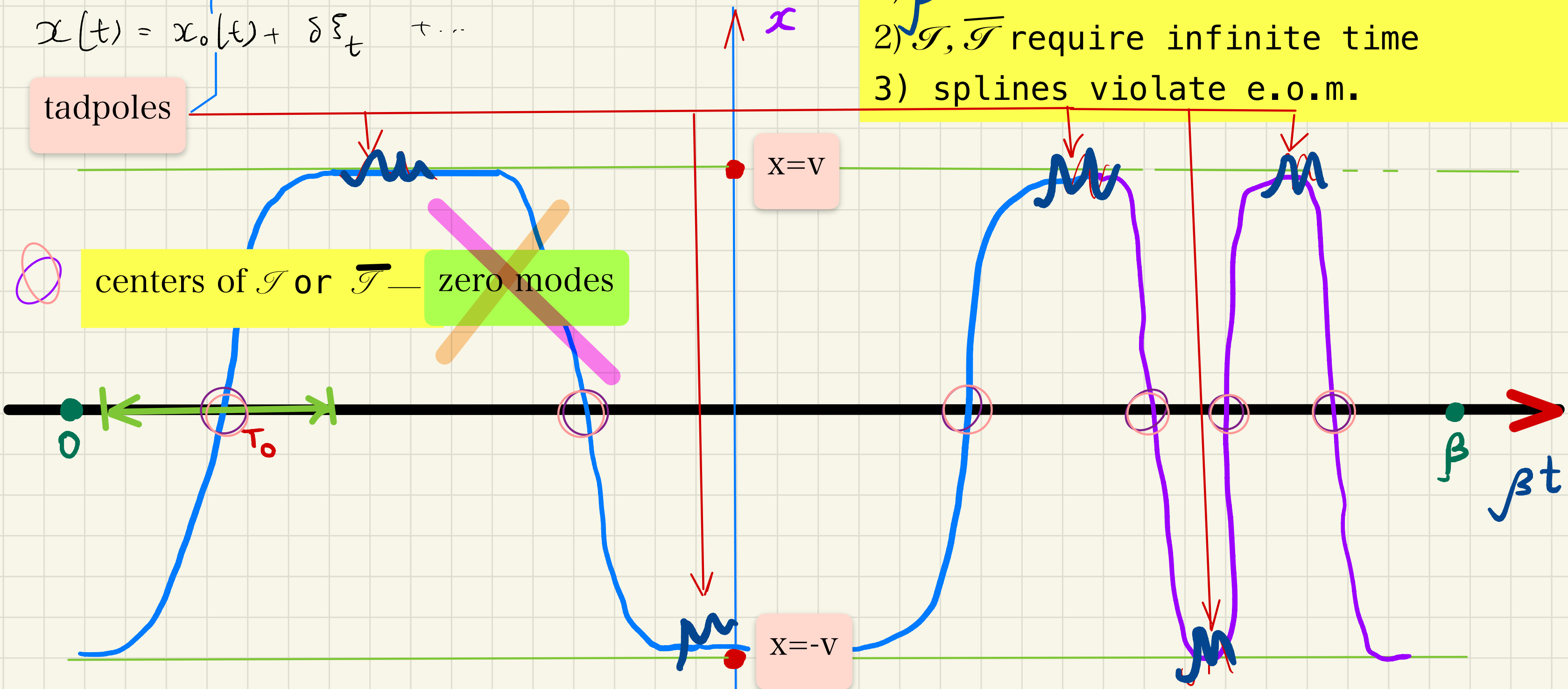
$$x(t) = x_0(t) + \delta\xi_t + \dots$$

Textbook calculation?

1) β does not have to be real

2) $\mathscr{I}, \bar{\mathscr{I}}$ require infinite time

3) splines violate e.o.m.

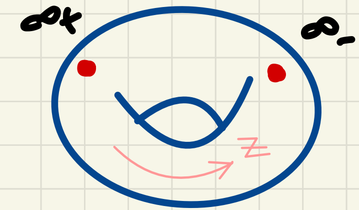


taking into account tadpoles

= interaction between $\mathscr{I}$ and $\bar{\mathscr{I}}$

$$T_0 \cong \frac{1}{\sqrt{U''_{xx}(v)}}$$

True picture of the critical points

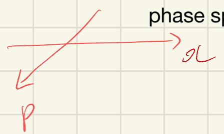


Elliptic curve

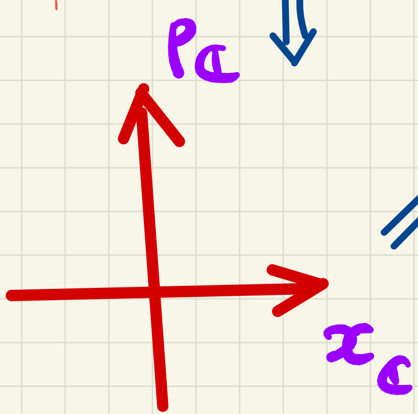
$$\begin{cases} \frac{i}{\hbar}\dot{x} = \beta \frac{\partial H}{\partial p} = \beta p \\ -\frac{i}{\hbar}\dot{p} = \beta \frac{\partial H}{\partial x} = \beta U'_x \end{cases}$$

$$\left\{ \frac{1}{2} p^2 + \frac{\lambda}{4}(x^2 - v^2)^2 = E \right\} = C_E$$

the critical orbit sticks out to the complex phase space



$\Downarrow$



$\nearrow$

$$H(p, x) = E$$

= constant in time

depends on the solution

complex

True picture of the critical points

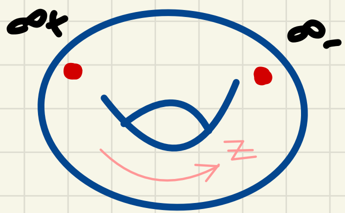
$$H(p,x) = E$$

= constant in time
depends on the solution

complex

$$\left\{ \frac{1}{2}p^2 + \frac{\lambda}{4}(x^2 - v^2)^2 = E \right\} = C_E$$

Elliptic curve



coordinated by

$$z = \int \frac{dx}{p} \sim z + p\,\omega_1 + q\,\omega_2$$

periods

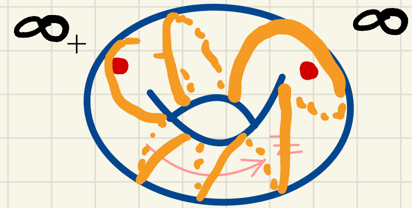
True picture of the critical points

$$H(p,x)=E$$

= constant in time
depends on the solution

periods

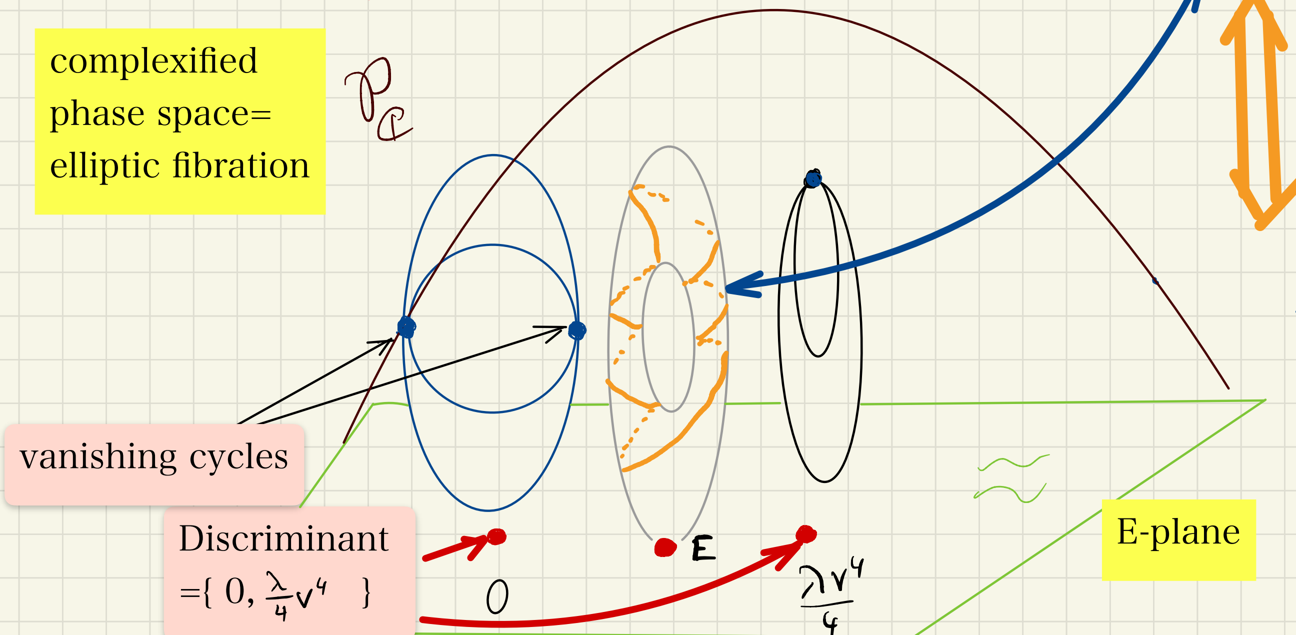
$$z=\int\frac{dx}{p}\sim z+m\,\omega_1(E)+n\,\omega_2(E)$$

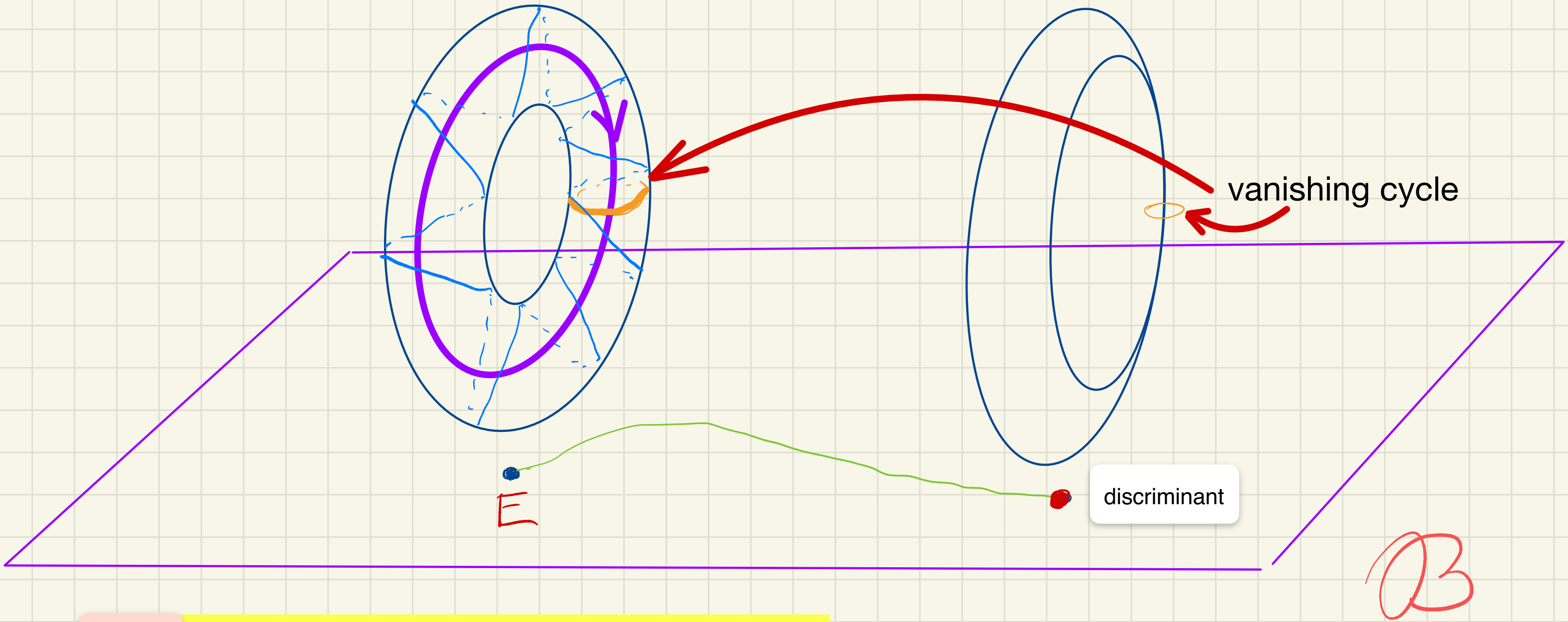


$\dot z=-i\hbar\beta$

periodic orbit (p,q)

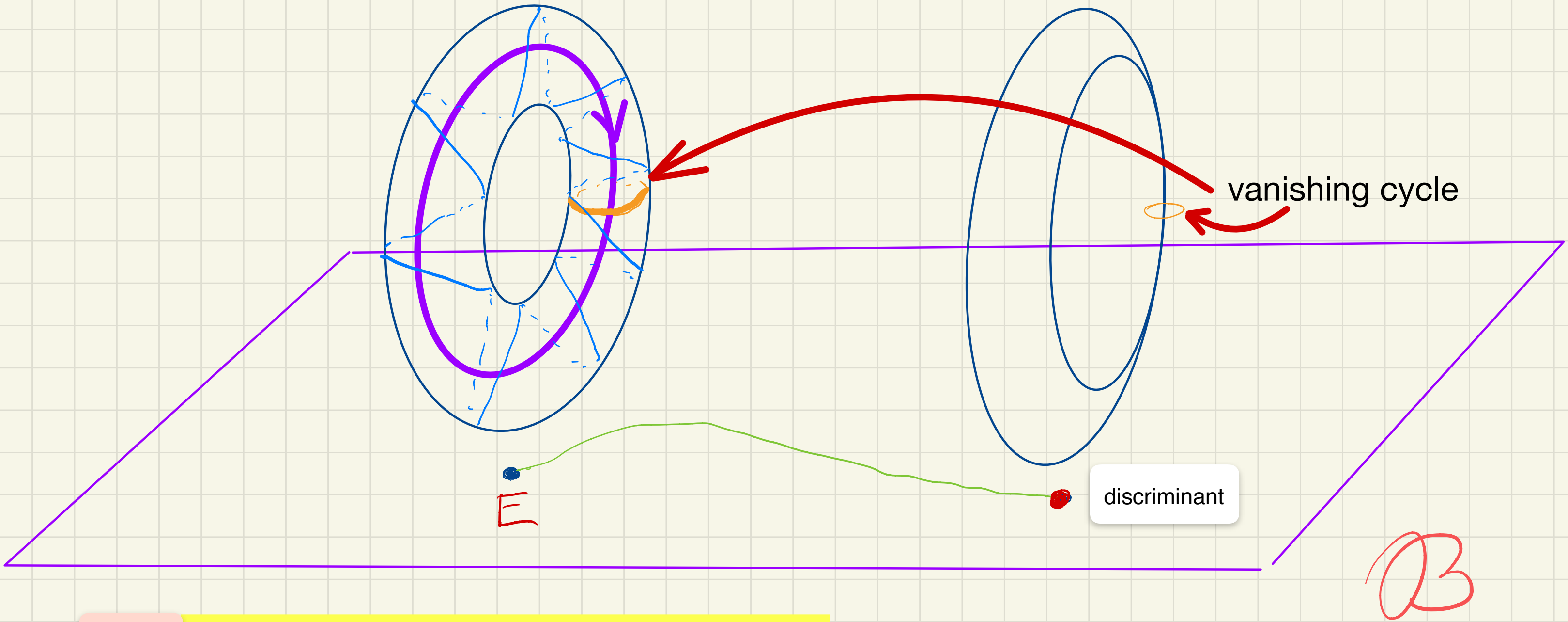
$$-i\beta\hbar = p\,\omega_1(E)+q\,\omega_2(E)$$





m = # periodic orbit $\cap$ vanishing cycle

n= defined modulo a multiple of m (Picard-Lefschetz monodromy)



m = # periodic orbit $\cap$ vanishing cycle

n= defined modulo a multiple of m (Picard-Lefschetz monodromy)

$\beta \to \infty$

$$E_{m,n} \cong \exp\left(\frac{2\pi i n}{m}\right) \exp\left(-\frac{\beta}{m T_0}\right) \times \frac{\lambda v^4}{4}$$

Solar system

mass scale

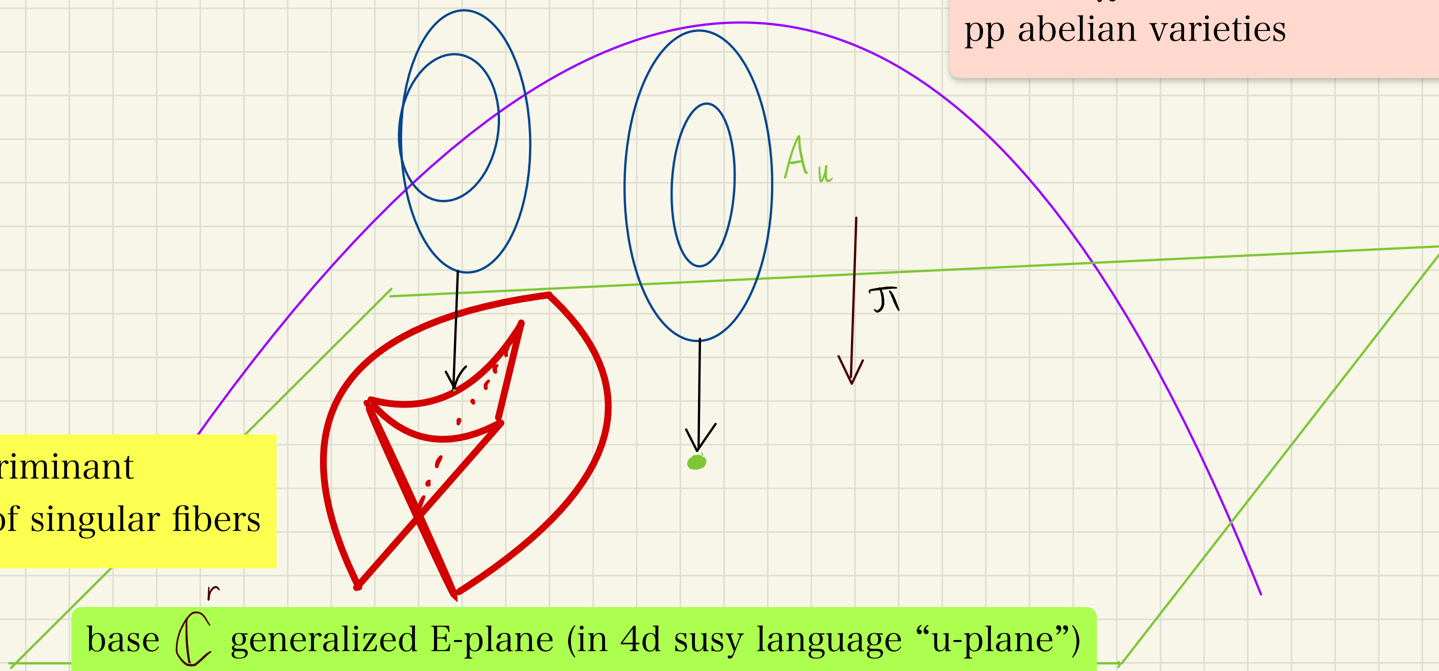
$$T_0 \cong \frac{1}{\sqrt{U''_{xx}(v)}}$$

Generalization to many degrees of freedom

$$(\mathcal{P}, \omega) \longrightarrow (\mathcal{P}^{\mathbb{C}}, \omega^{\mathbb{C}})$$

$$\pi(p, x) = (u_1, \ldots, u_r)$$

lagrangian projection
fibers $A_u = \pi^{-1}(u)$
pp abelian varieties



discriminant
set of singular fibers

base $\mathbb{C}^r$ generalized E-plane (in 4d susy language “u-plane”)

Generalization to many degrees of freedom

$$(\mathcal{P},\omega) \longrightarrow (\mathcal{P}^{\mathbb{C}},\omega^{\mathbb{C}})$$

$$\pi(p,x) = (u_1,\dots,u_r)$$

lagrangian projection

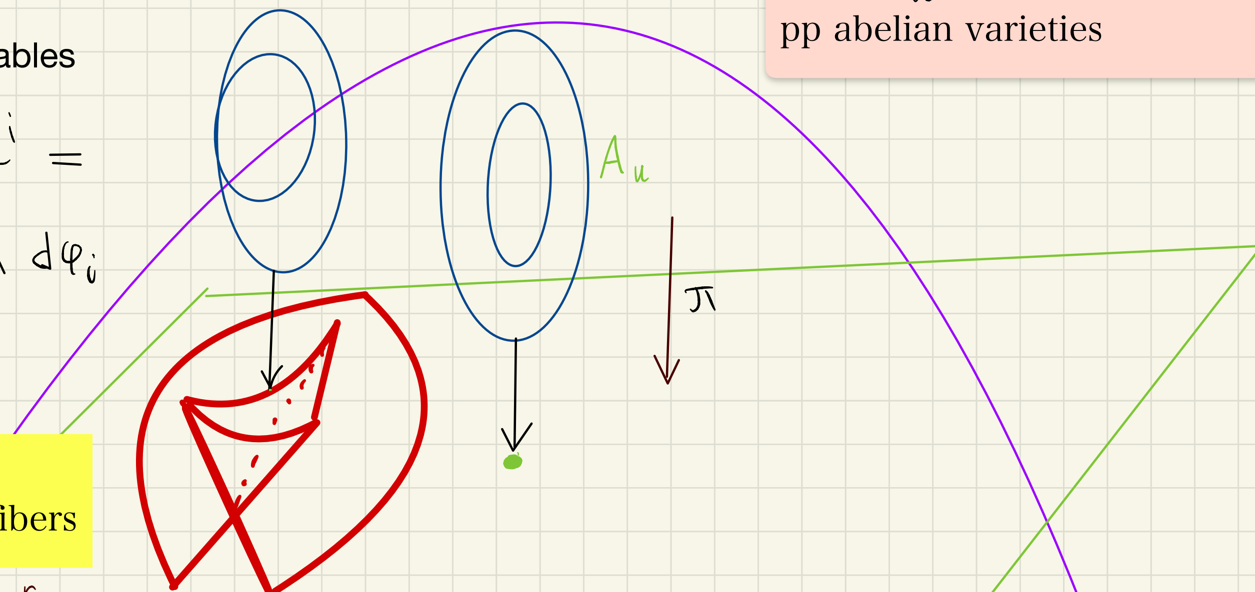
fibers $A_u = \pi^{-1}(u)$

pp abelian varieties

action-angles variables

$$\omega^{\mathbb{C}} = \sum_i dp_i \wedge dx^i = \sum_i da^i \wedge d\varphi_i$$

$$\frac{\partial u_\ell}{\partial \varphi_i} = 0$$



discriminant

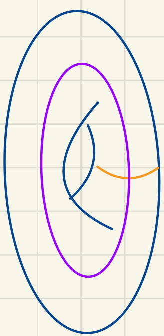
set of singular fibers

base $\mathbb{C}^r$ generalized E-plane (in 4d susy language “u-plane”)

$$a^i(t) = a^i(0) \qquad i=1,\dots,r$$

$$\varphi_i(t) = \varphi_i(0) + \frac{\beta\hbar}{i}\frac{\partial H}{\partial a^i}\, t \,, \qquad 0 \le t \le 1$$

periodic orbit =



$$\varphi_i(1) - \varphi_i(0) = 2\pi\left(n_i + \tau_{ij} m^j\right) = \frac{\beta\hbar}{i}\frac{\partial H}{\partial a^i}$$

$$\vec{n}, \vec{m} \in \mathbb{Z}^r$$

period matrix

$$\boxed{\frac{\partial a_{i,D}}{\partial a^j} = \tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}}$$

prepotential of special geometry

$$a^i = \frac{1}{2\pi}\oint_{A_i} p\,dx$$

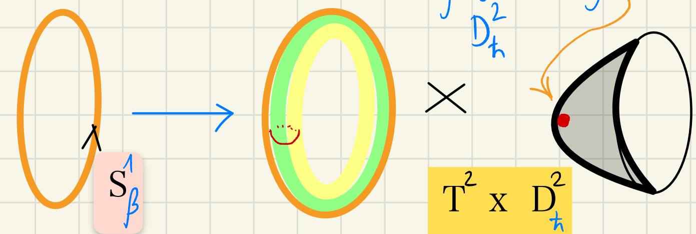
$$a_{D,i} = \frac{1}{2\pi}\oint_{B^i} p\,dx$$

quantum mechanics 1d $\longleftrightarrow$ related to d=4 N=2 super-Yang-Mills

$$\boxed{\frac{\beta\hbar}{2\pi i}\frac{\partial H}{\partial a^k} = n_k + \tau_{k\ell}\, m^\ell}$$

equation on $\vec{u}$ given $(\vec{n},\vec{m}) \in \mathbb{Z}^r$

$\beta\hbar\int_{D^2_\hbar} H^{(2)} \sim \beta H$



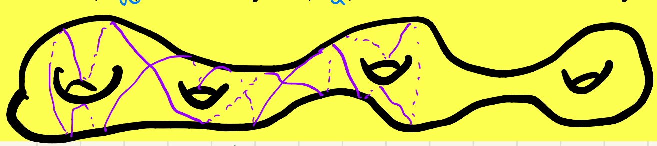
S^1_β $\longrightarrow$ $T^2 \times D^2_\hbar$

$$\boxed{d W_\gamma(\vec{u}) = 0}$$

$$\gamma = n_k A^k + m^\ell B_\ell \in H_1(A_{u_0}, \mathbb{Z})$$

$$\boxed{W_\gamma(\vec{u}) = \frac{i}{\hbar}\oint_\gamma d^{-1}\omega_{\mathbb{C}} - \beta H(\vec{u})}$$

Quite often the abelian variety is Jac(C_u), or Prym(C_u), for some family of **spectral curves** C_u



$$\gamma \in H_1(C_{u_0}, \mathbb{Z})$$

Field theory : 1+1 dimensional case

fields

$$\phi : \Sigma \longrightarrow X$$

$$S = \int_{\Sigma} G_{\mu\nu}\, \partial X^{\mu}\, \bar{\partial} X^{\nu}$$

Σ Riemann surface

Configuration space

Maps(Σ , X)

$$X \xrightarrow{1} S^{N-1}$$
$$X \xrightarrow{2} \mathbb{CP}^{N-1}$$

$\vec{x} \in \mathbb{R}^N$

1) $\sum_{i=1}^{N} x_i^2 = 1$

2) $\sum_{i=1}^{N} z_i \bar{z}_i = 1$

$\vec{z} \in \mathbb{C}^N$

modulo U(1) action

$z \mapsto e^{i\theta} z, \quad \bar{z} \mapsto e^{-i\theta} \bar{z}$

Field theory : 1+1 dimensional case

fields $\qquad \phi : \Sigma \longrightarrow \mathcal{X}$

$$S = \int_\Sigma G_{\mu\nu}\, \partial X^\mu\, \bar{\partial} X^\nu$$

Σ Riemann surface

Configuration space $\mathbb{C}$

Maps(Σ , $\mathcal{X}^{\mathbb{C}}$)

$$\mathcal{X} \xrightarrow{1} S^{N-1}$$

$$\mathcal{X} \xrightarrow{2} \mathbb{C}P^{N-1}$$

$\vec{x} \in \mathbb{C}^N$

1) $\displaystyle\sum_{i=1}^{N} x_i^2 = 1$

2) $\displaystyle\sum_{i=1}^{N} z_i \bar{z}_i = 1$

modulo $\mathbb{C}^\times$ action

$z \mapsto t\, z, \quad \bar{z} \mapsto t^{-1}\, \bar{z}$

independent z & $\bar{z}$

$\vec{z} \in \mathbb{C}^N$

$\vec{\bar{z}} \in \mathbb{C}^N$

Twisted partition function

$$\mathrm{Tr}_{\mathcal{H}_h}\left(\exp(-\beta\hat{H})\,\exp(i\vartheta\hat{P})\; g\right)$$

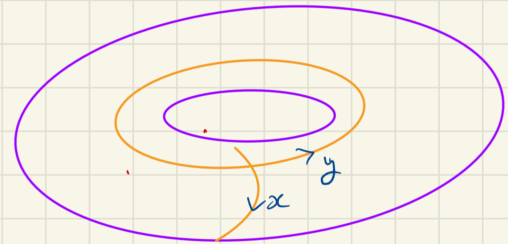
$$gh = hg \,, \quad g, h \in O(N)$$

$\hat{P} = -i\partial_x$ space translations

$\hat{H} = -\cancel{i}\partial_y$ time evolution

euclidean

background flat O(N) - connection



$$ds^2 = \left(dx + \frac{\vartheta}{2\pi}dy\right)^2 + \beta^2 dy^2$$

x~x+1, y~y+1

$\beta, \vartheta \in \mathbb{C}$

TWISTS

$\vec{\Psi} = (Z, \tilde{Z})$ $\quad Z \in \mathbb{C}^m, \tilde{Z} \in \mathbb{C}^m$ $\quad N = 2m$ case $\quad O(2m, \mathbb{C})$

$$\boxed{\begin{aligned} Z^a(x+1, y) &= h_a Z^a(x,y) & Z^a(x, y+1) &= g_a Z^a(x,y) \\ \tilde{Z}_a(x+1, y) &= h_a^{-1} \tilde{Z}_a(x,y) & \tilde{Z}_a(x, y+1) &= g_a^{-1} \tilde{Z}_a(x,y) \end{aligned}}$$

Complex metric, complex twists

$$S = \frac{1}{\tau - \bar{\tau}} \int_{T^2} dx dy \left\{ \partial_y Z \partial_y \tilde{Z} - \frac{\vartheta}{2\pi} \partial_y Z \partial_x \tilde{Z} - \frac{\vartheta}{2\pi} \partial_x Z \partial_y \tilde{Z} + \tau\bar{\tau} \partial_x Z \partial_x \tilde{Z} \right\}$$
$$+ u(\tilde{Z} \cdot Z - 1)$$

$\tau = \frac{\vartheta}{2\pi} + i\beta$

$\bar{\tau} = \frac{\vartheta}{2\pi} - i\beta$

two independent complex parameters

$\partial = \frac{\bar{\tau}\partial_x - \partial_y}{\bar{\tau} - \tau}$

$\bar{\partial} = \frac{\tau\partial_x - \partial_y}{\tau - \bar{\tau}}$

Equations of motion

$$\partial\bar{\partial}\vec{\Psi} = u \vec{\Psi}$$

$$u = -\partial\vec{\Psi} \cdot \bar{\partial}\vec{\Psi}$$

Self-consistent Schrödinger potentials on T^2

$$u(z, \bar{z}) = u(z+1, \bar{z}+1) = u(z+\tau, \bar{z}+\bar{\tau})$$

Given complex Schrödinger potential $u(z,\bar{z})$, define the Bloch set $\mathcal{B}_u$ and Fermi curve $\mathcal{C}_u$

$$\mathcal{B}_u = \left\{ (a,b) \;\middle|\; \exists\, \psi,\quad \partial\bar{\partial}\,\psi = u(z,\bar{z})\,\psi, \quad \begin{array}{l} \psi(z+1,\bar{z}+1) = a\,\psi(z,\bar{z}) \\ \psi(z+\tau,\bar{z}+\bar{\tau}) = b\,\psi(z,\bar{z}) \end{array} \right\}$$

$\mathcal{C}_u$ = "normalization of the Bloch set"

To get a taste of the problem consider the case u= const = u_0

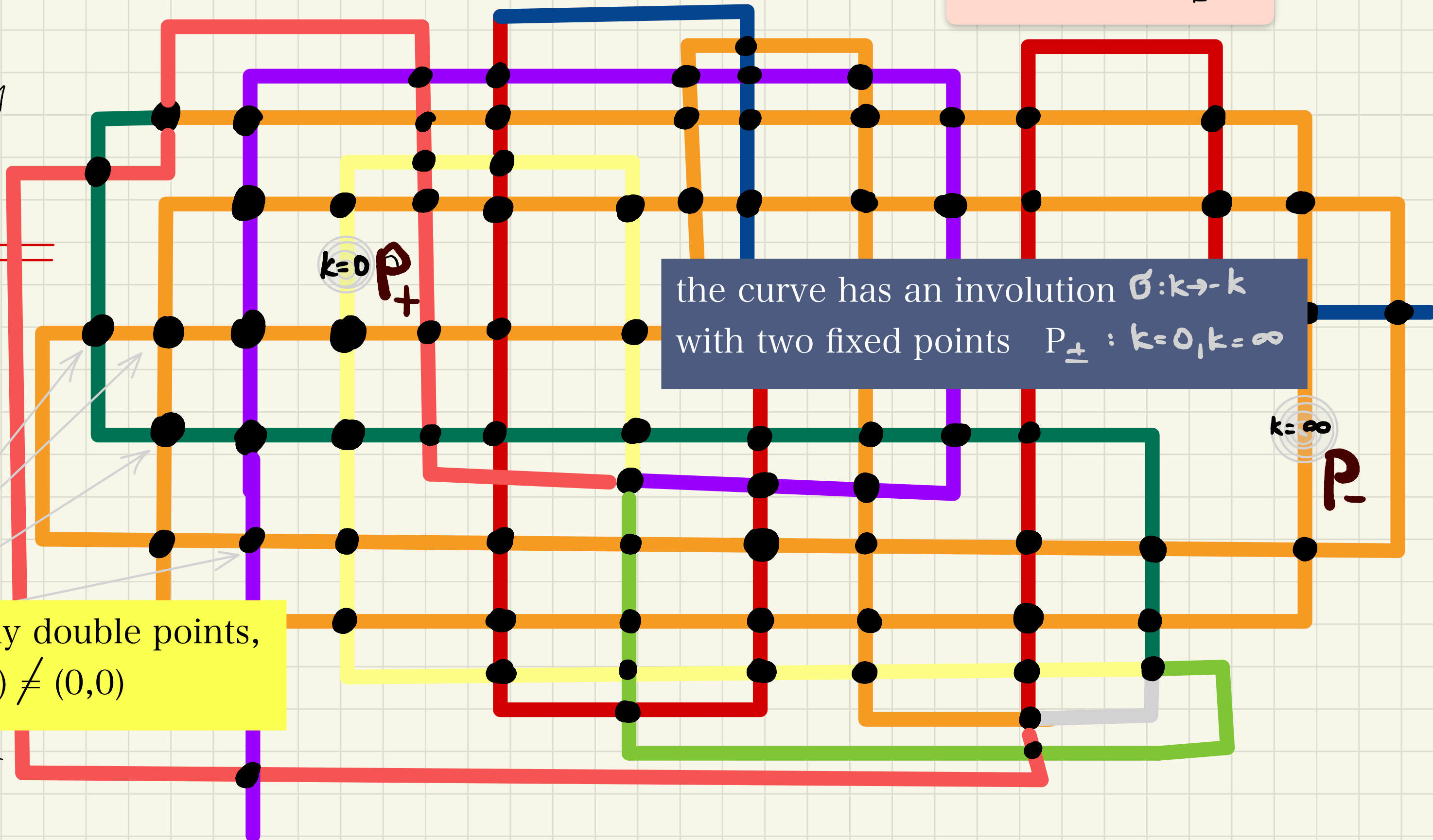
$$\partial\bar\partial \Psi = u_0 \Psi \Rightarrow \Psi = \exp\left(k z + \frac{u_0}{k}\bar z\right) \Rightarrow \begin{array}{l} a = \exp\left(k + \frac{u_0}{k}\right) \\ b = \exp\left(k\tau + \frac{u_0}{k}\bar\tau\right) \end{array}$$

$C_{u_0} \ni k$

$\approx \mathbb{P}^1$

$\searrow$

$B_{u_0} =$



the curve has an involution $\sigma : k \to -k$
with two fixed points $P_\pm$: $k=0, k=\infty$

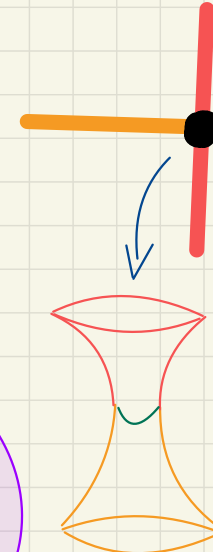
infinitely many double points,
for each $(m,n) \neq (0,0)$
$\in \mathbb{Z}^2$

$$\partial\bar{\partial}\Psi = u\Psi$$

$$u(z,\bar{z}) = u_0 + \sum_{(m,n)\in\mathbb{Z}^2\setminus 0} u_{m,n}\, e^{2\pi i (mx+ny)}$$

$z = x + \tau y$

$\bar{z} = x + \bar{\tau} y$

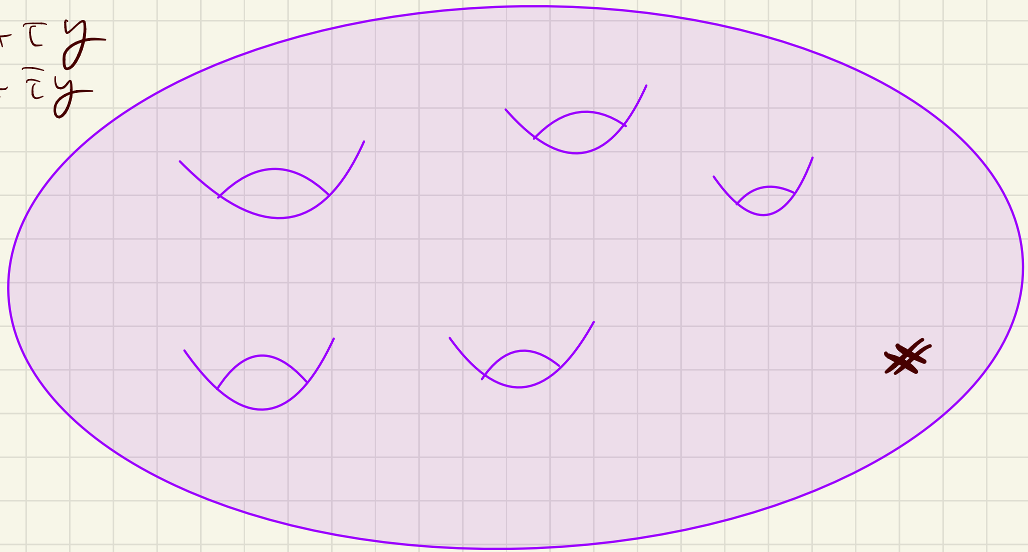


$zw = 0$

(m,n) point

$zw \approx$

$\approx u_{m,n}\, u_{-m,-n}$



algebra-geometric $u(z,\bar{z}) = C_u$ is algebraic finite genus curve

The set of algebraic geometric potentials is rare but dense among all Schrödinger potentials

We found a characterization of potentials u

for which $(\partial\bar\partial - u)\psi_i = 0$, $-u = \sum_{i=1}^{N} \partial\psi_i \bar\partial\psi_i$

algebra-geometric data

$\sigma : C_u \circlearrowleft \quad \sigma^2 = id$

$\{P_+, P_-\} = C_u^\sigma$

$C_u \supset \{\gamma_1^\pm, \dots, \gamma_g^\pm\}$

$E : C_u/\sigma \longrightarrow P^1$

isolated solutions, discrete set

$\Longrightarrow$

$(\psi_i)_{i=1}^{N} = \{\psi(q_i) \mid q_i, \sigma(q_i)\}$

$E^{-1}(\infty)$

$\sigma(\gamma_i^\pm) = \gamma_i^\mp$

$$\psi = e^{z\Omega_- + \bar z\Omega_+} \prod \frac{\vartheta(\ )}{\vartheta(\ )}$$

Baker-Akhiezer function

$u = \partial\bar\partial \log \vartheta(\dots)$

Analogy: twistor and spectral curves for monopoles

$D_A\Phi = {}^*F_A$

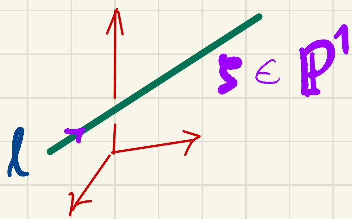
Bogomolny equation

on $\mathbb{R}^3$

$\mathcal{C} \subset T\mathbb{P}^1$

twistor curve

$\Rightarrow$ réduction to principal chiral field spectral curves



$\ell \in \mathcal{C} \Leftrightarrow \nabla_A^\ell + \Phi$ has L^2 zero mode

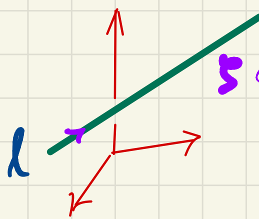
Analogy: twistor and spectral curves for monopoles

$D_A\Phi = {}^*F_A$

Bogomolny equation

on $\mathbb{R}^2 \times S^1$ ① $\mathcal{C} \subset T\mathbb{P}^1/\mathbb{Z}$

twistor curve



$\zeta \in \mathbb{P}^1$

$\ell \in \mathcal{C} \leftrightarrow \nabla_A^{\ell} + \Phi$ has L^2 zero mode

② spectral curve

$C \subset \underset{x}{\mathbb{C}} \times \underset{w}{\mathbb{C}^\times}$

$g(x) = P\exp \oint_{S^1} A + i\Phi$

Det(g(x) -w)=0

Tantalizing example in four dimensional Yang-Mills theory

$$\int \mathrm{Tr}\, F_A \wedge {}^* F_A$$

$$= \int \dot{f}^2 dt \; \|\Theta\|^2_{L^2_{S^3}}$$

$$+ \int f^2 (1-f)^2 dt \; \|\Theta \wedge \Theta\|^2_{L^2_{S^3}}$$

$$M^4 = \mathbb{R}^1 \times S^3$$

time t space

fixed

$g : S^3 \to G$

Ansatz

homolorphism

$$A = f(t) \; g^{-1} dg$$

$$F_A = \dot{f}\, dt \wedge g^{-1} dg + (f^2 - f)(g^{-1} dg)^2$$

$$\frac{1}{8\pi^2} \mathrm{Tr}\, F_A \wedge F_A = \frac{1}{8\pi^2} \mathrm{Tr}(g^{-1} dg)^3 \wedge d\left(\frac{f^3}{3} - \frac{f^2}{2} \right)$$

$$\frac{1}{8\pi^2} \int \mathrm{Tr}\, F_A \wedge F_A = k \left(\Delta f^3 - \frac{3}{2} \Delta f^2 \right)$$

$f, \dot{f}$