Overlaps and Fermionic Dualities for Integrable Super Spin Chains

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Based on:

- C.K., D. Müller & K. Zarembo, ArXiv:2005.01392[hep-th], JHEP 08 (2020) 103
- C.K., D. Müller & K. Zarembo, ArXiv:2011.12192[hep-th], to appear in JHEP
- Work in progress involving also A. Wallberg (BSc.) and M. Lauritzen (MSc.)

LIJC
Jan. 21st, 2021

Motivation

- An integrable psu(2,2|4) super spin chain underlies AdS/CFT
- Overlaps between eigenstates and Matrix Product States or Valence Bond States encode information about one-point fcts. in AdS/dCFT (and about quantum quenches in stat. mech.)
- Fermionic dualities allow one to move between different Dynkin diagrams of the underlying super Lie algebra
- Fermionic dualities allow to translate overlap formulas from one Dynkin diagram to another (and allow comparison).
- Fermionic dualities constrain overlap formulas

Plan of the talk

I. Integrable overlaps in AdS/dCFT

II. The Structure of overlap formulas

III. Fermionic Dualities

IV. Translating overlap formulas between different Dynkin diagrams

V. Outlook

Integrable overlaps in AdS/dCFT

Bethe eigenstate

$\langle B|\mathbf{u}\rangle$ computable in closed form

Matrix product states

$$|B\rangle = |\mathrm{MPS}\rangle = \sum_{\{s_i\}} \mathrm{Tr}(t_{s_1}\ldots t_{s_L})|s_1\ldots s_L\rangle$$

Valence Bond States

$$|\mathrm{VBS}\rangle = |K\rangle^{\otimes \frac{L}{2}}, \qquad K = \sum_{s_1,s_2} K_{s_1,s_2}|s_1 s_2\rangle$$

Integrability understood in a scattering picture

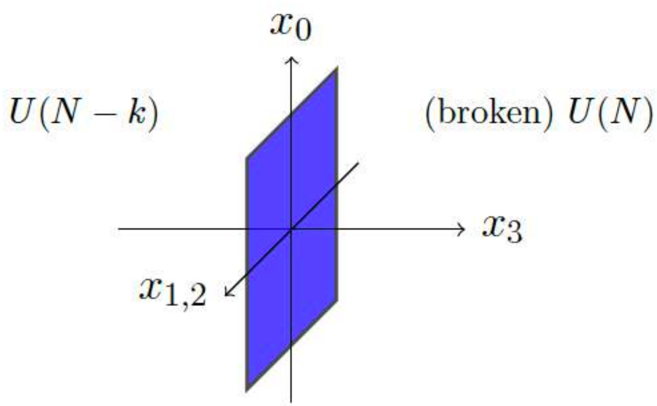
$$\mathbf{Q}_{2n+1}|B\rangle = 0$$

Charge

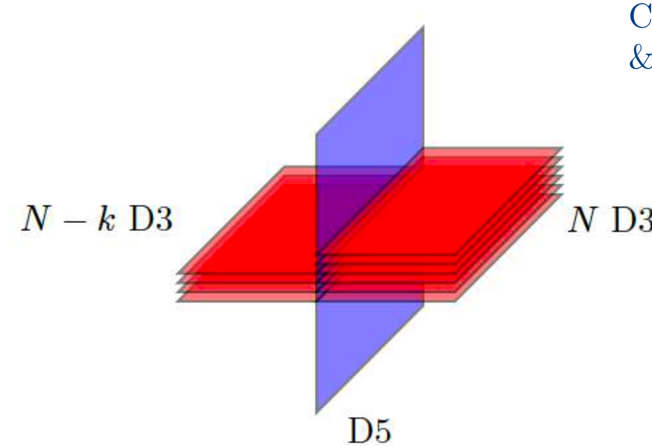
Ghoshal & Zamolodchikov '94

Piroli, Pozsgay Vernier '17

AdS/dCFT set-up



Gauge Theory



String Theory

Constable, Myers & Tafjord '99

For $x_3 > 0$: $$\phi_i^{\text{cl}} = \frac{1}{x_3}\begin{pmatrix}(t_i)_{k\times k} & 0\\ 0 & 0\end{pmatrix}, \quad i = 1,2,3$$

where t_i, $i = 1,2,3$ constitute a k-dimensional irreducible representation of $\mathfrak{su}(2)$

Set-up 1/2 BPS Gaiotto & Witten '08

Perturbative computations

Buhl-Mortensen, de Leeuw, Ipsen, C.K, Wilhelm '16

For $x_3 > 0$: Expanding around the classical solution

$$\text{Quantum fields } A_\mu, \Phi_i, \Psi = \begin{array}{c} \quad k \qquad N-k \\ \left[\begin{array}{c|ccc} x & y & y & y \\ \hline y & z & z & z \\ y & z & z & z \\ y & z & z & z \end{array}\right] \end{array} \begin{array}{c} k \\ \\ N-k \end{array}$$

For k>1: x and y fields are massive, $m^2 \propto 1/x_3^2$, have AdS propagators.
(except gauge field),
z fields are massless

For k=1: No vevs

	$\Phi_{4,5,6}, A_{0,1,2}, c$	$\Phi_{1,2,3}, A_3$
x, y	Dirichlet	Neumann
z	no BCs	no BCs

One-point functions in dCFT's

$$\langle \mathcal{O}^{\text{bulk}}_{\Delta}(x) \rangle = \frac{C}{|x_3|^{\Delta}}$$

Cardy ´84
McAvity & Osborn '95

k> 1:

Due to vevs scalar operators can have non-zero 1-pt fcts at tree-level

$$\langle \mathcal{O}_{\Delta}(x) \rangle = \left(\text{Tr}(\phi_{i_1} \ldots \phi_{i_\Delta}) + \ldots \right) \Big|_{\phi_i \to \phi_i^{\text{cl}} = \frac{t_i}{x_3}}$$

Matrix Product State associated with the defect:

deLeeuw, C.K. & Zarembo '15,

$$|\text{MPS}_{\text{k}}\rangle = \sum_{\vec{i}} \text{tr}[t_{i_i} \ldots t_{i_L}] |\phi_{i_1} \ldots \phi_{i_L}\rangle,$$

Object to calculate:

$$C_k(\mathbf{u}) = \frac{\langle \text{MPS}_k | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle^{\frac{1}{2}}}$$

The k=1 case

Propagators for complex scalars: $X = \phi_1 + i\phi_4$, etc.

$$D_\kappa(x,y) = \frac{1}{4\pi^2}\left(\frac{1}{|x-y|^2} + \frac{\kappa}{|\bar{x}-y|^2}\right), \quad \kappa = \begin{cases} 1 & \text{Neumann} \\ -1 & \text{Dirichlet} \\ 0 & \text{no BCs.} \end{cases}$$

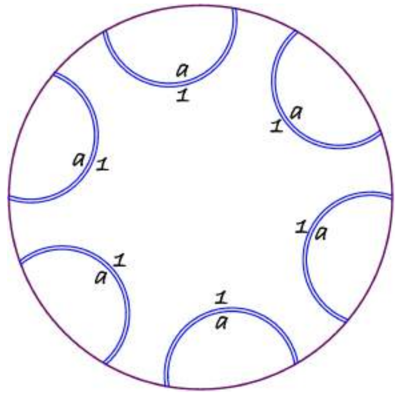
$$\bar{x} = (x_0, x_1, x_2, -x_3)$$

$$\langle X^{1a}(x) X^{b1}(y)\rangle = \frac{g_{\rm YM}^2 \delta^{ab}}{2}\Big(D_1(x,y) - D_{-1}(x,y)\Big) = \frac{g_{\rm YM}^2 \delta^{ab}}{4\pi^2 |\bar{x}-y|^2}\,,$$

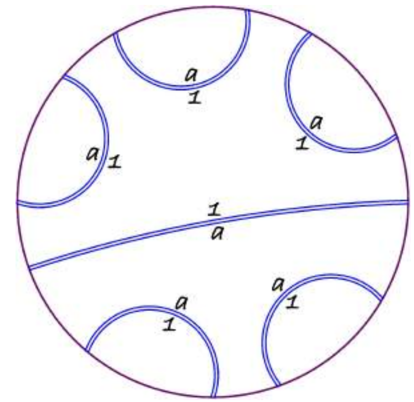
Propagators for fermions in the SU(2|3) sector

$$\langle \Psi_\alpha^{1a}(x) \Psi_\beta^{b1}(y)\rangle = \frac{g_{\rm YM}^2}{8\pi^2}\,\epsilon_{\alpha\beta}\,\delta^{ab}\;\cdot\;\frac{\bar{x}_3 - y_3}{|\bar{x}-y|^4}.$$

Feynman diagrams



Leading for large-N



Sub-leading for large-N

Object to calculate $C_{k=1} = 2\left(\frac{g_{\rm YM}^2 N}{16\pi^2}\right)^{L/2} \frac{\langle {\rm VBS}|\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle^{1/2}}$

C.K., Müller, Zarembo '20

$\langle {\rm VBS}| = (\langle XX| + \langle YY|)^{\otimes L/2}, \qquad SU(2)$ sector

$\langle {\rm VBS}| = (\langle XX| + \langle YY| + \langle ZZ| + \langle \uparrow\downarrow| - \langle \downarrow\uparrow|)^{\otimes L/2}, \qquad SU(2|3)$ sector

$|{\rm VBS}\rangle$'s also of importance as initial steps in proofs of overlap formulas for $|{\rm MPS}\rangle$'s

de Leeuw, Gombor, C.K., Linardopoulos, Pozsgay '19

Gombor & Bajnok '20

Overlap Formulas

Ingredients:

For $|\mathrm{MPS}_k\rangle$: de Leeuw, C.K. & Zarembo '15 de Leeuw, C.K. & Mori '16 de Leeuw, C.K. & Linardopoulos '18 de Leeuw, Gombor, C.K., Linardopoulos, Pozsgay '19

- Superdeterminant of Gaudin matrix G: $\langle \mathbf{u}|\mathbf{u}\rangle \propto \det(G)$
- Ratios of Baxter polynomiums (reduced): $Q(u) = \prod_i (u^2 - u_i^2)$
- "Transfer matrices": Sums of ratios of Baxter polynomials: $\sum_{a=-\frac{k}{2}}^{a=\frac{k}{2}} \ldots$

For $|\mathrm{VBS}\rangle$:

- No sums involved Poszgay '18

Will focus on $|\mathrm{VBS}\rangle$ of relevance for the $k = 1$ case in AdS/dCFT

- Understand the structure of overlap formulas for super spin chains
- Investigate their transformation properties under fermionic dualities (change of Dynkin diagram)

Examples

$$SU(2): \quad |\mathrm{VBS}\rangle = (|XX\rangle + |YY\rangle)^{\otimes L/2}, \quad C^2 = \frac{Q(0)}{Q(\frac{i}{2})} S \det G$$

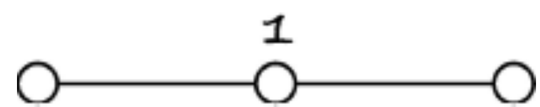
Poszgay '18

$$SU(3): \quad |\mathrm{VBS}\rangle = |XX\rangle + |YY\rangle + |ZZ\rangle \ , \quad C = \frac{Q_1(0)Q_2(0)}{Q_1(\frac{i}{2})Q_2(\frac{i}{2})} S\mathrm{det}G$$

Piroli, Vernier, Calabrese, Poszgay '19

$$SO(6): \quad |\mathrm{VBS}\rangle = |XX\rangle + |YY\rangle + |ZZ\rangle + |\bar{X}\bar{X}\rangle + |\bar{Y}\bar{Y}\rangle + |\bar{Z}\bar{Z}\rangle \ ,$$

$$C = \frac{Q_1(0)Q_2(0)Q_3(0)}{Q_1\left(\frac{i}{2}\right)Q_2\left(\frac{i}{2}\right)Q_3\left(\frac{i}{2}\right)} S\mathrm{det}G$$



de Leeuw, Gombor, C.K., Linardopoulos, Pozsgay '19

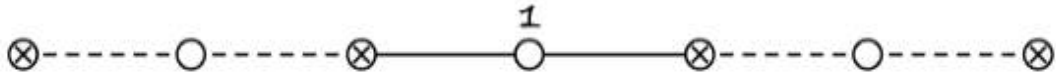
$$SU(2|1): \quad |\mathrm{VBS}\rangle = |XX\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

$$C = \frac{Q_1(0)}{Q_2(0)Q_2\left(\frac{i}{2}\right)} S\mathrm{det}G$$

$SU(2)$ matches analytically continued (dressed, higher loop) $k > 1$ formula

The $\mathcal{N}=4$ SYM spin chain

$$PSU(2,2|4): \quad C = \frac{Q_1(0)Q_3(0)Q_4(0)Q_5(0)Q_7(0)}{Q_2(0)Q_2(\frac{i}{2})Q_4(\frac{i}{2})Q_6(0)Q_6(\frac{i}{2})} S \det G$$

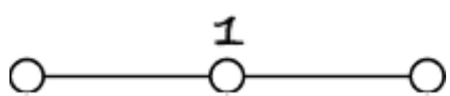


Analytical continuation of (higher loop) result for $|\mathrm{MPS}_k\rangle$ from

Bajnok & Gombor '20

See also Komatsu & Wang '20

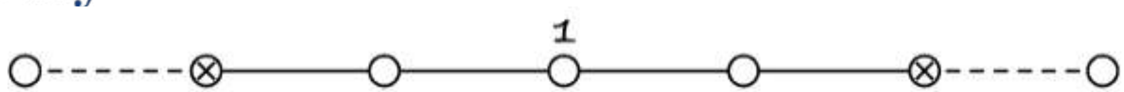
- How to compare to previous results ?



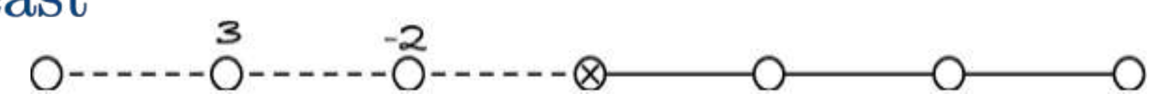
- What does the expression look like for other gradings ?

Beisert & Staudacher '04

  – The Beauty

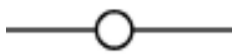


  – The Beast



Integrable Super Spin Chains (of type $GL(M|N)$)

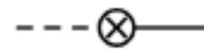
Cartan matrix: M_{ab}, Dynkin labels q_a, $a,b=1,\dots n$



$M_{aa}=2,$
$M_{aa+1}=-1,$
$M_{aa-1}=-1.$

$M_{aa}=-2,$
$M_{aa+1}=+1,$
$M_{aa-1}=+1.$

$M_{aa}=0,$
$M_{aa+1}=+1,$
$M_{aa-1}=-1.$



$M_{aa}=0,$
$M_{aa+1}=-1,$
$M_{aa-1}=+1.$

Bethe equations

$$\left(\frac{u_{a,j}-\frac{iq_a}{2}}{u_{a,j}+\frac{iq_a}{2}}\right)^L \prod_{b,k} \frac{u_{a,j}-u_{b,k}+\frac{iM_{ab}}{2}}{u_{a,j}-u_{b,k}-\frac{iM_{ab}}{2}} \equiv \mathrm{e}^{\,i\chi_{a,j}} = (-1)^{q_a}$$

$a=1,\dots n$ (# of nodes), $j=1,\dots,K_a$ (# of roots of type a)

Gaudin matrix $\quad G_{aj,bk}=\dfrac{\partial\chi_{aj}}{\partial u_{bk}}$ $\quad$ of size $\sum_a K_a \times \sum_a K_a$

Integrable overlaps and the Gaudin determinant

$$\hat{\mathbf{Q}}_{\mathbf{2n+1}}|B\rangle = 0 \implies$$

$\langle B|\mathbf{u}\rangle \neq 0$ iff momentum carrying roots are paired $\{u_i, -u_i\}_{i=1}^{K_u}$
(excluding singular cases)

$\implies$ auxiliary roots paired $\{v_i, -v_i\}_{i=1}^{K_v}$ possibly plus $\{0\}$

Gaudin matrix has block structure

$$\det G = \begin{vmatrix} A & B \\ B & A \end{vmatrix} = \begin{vmatrix} A+B & B \\ B+A & A \end{vmatrix} = \begin{vmatrix} A+B & B \\ 0 & A-B \end{vmatrix} = \det(A+B)\cdot\det(A-B)$$

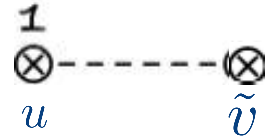
$$= \det G_+ \cdot \det G_-$$

Quantity entering overlap formulas

$$\mathrm{SDet}\, G = \frac{\det G_+}{\det G_-} \equiv \mathbb{D}$$

Fermionic Duality: Ex: $SU(2|1)$

$$M=\begin{bmatrix}2 & -1\\ -1 & 0\end{bmatrix},\ q=\begin{bmatrix}1\\0\end{bmatrix}$$



$$\widetilde{M}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ \tilde{q}=\begin{bmatrix}1\\0\end{bmatrix}$$

$$1=\prod_{n=1}^{K_1}\frac{v_k-u_n-\frac{i}{2}}{v_k-u_n+\frac{i}{2}}=\frac{Q_1^-(v_k)}{Q_1^+(v_k)}\quad\longrightarrow\quad\frac{Q_1^+(\tilde{v}_k)}{Q_1^-(\tilde{v}_k)}=1$$

$$-1=\frac{Q_1^{++}(u_k)}{Q_1^{--}(u_k)}\cdot\frac{Q_2^-(u_k)}{Q_2^+(u_k)}\left(\frac{u_k-\frac{i}{2}}{u_k+\frac{i}{2}}\right)^L\quad\longrightarrow\quad\frac{\widetilde{Q}_2^+(u_k)}{\widetilde{Q}_2^-(u_k)}\left(\frac{u_k-\frac{i}{2}}{u_k+\frac{i}{2}}\right)^L=1$$

Change of variables

$\widetilde{K}_2=K_1-K_2-1$ roots $\tilde{v}_i$

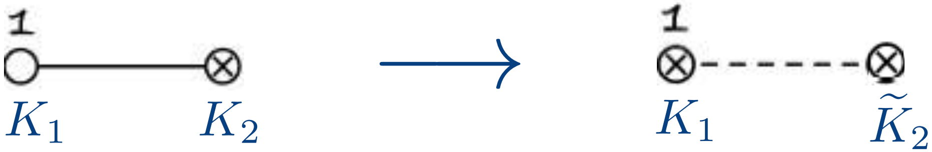
$$Q_1^-(v)-Q_1^+(v)=Q_2(v)\cdot\widetilde{Q}_2(v)$$

K_2 roots v_i

$$\frac{Q_1^{++}}{Q_1^{--}}=\frac{Q_2^+}{Q_2^-}\frac{\widetilde{Q}_2^+}{\widetilde{Q}_2^-}$$

$$Q^\pm(u)\equiv Q(u\pm\frac{i}{2})$$

Transformation formula: Ex: $SU(2|1)$



K_1, K_2 even $\implies$ $\widetilde{K_2} = K_2 - K_1 - 1$ odd, i.e. $\tilde{v}$'s contain a single zero

$Q_1^-(u) - Q_1^+(u) = u\, Q_2(u)\, \widetilde{Q}_2(u)$, with reduced Baxter polynomials

$$\widetilde{\mathbb{D}} = K_1 \frac{\widetilde{Q}_2(0) Q_2(0)}{Q_1(\frac{i}{2})} \mathbb{D}$$

Found numerically C.K., Müller, Zarembo '20

Analytical proof in progress

Notice:

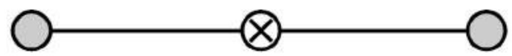
- Holds semi-on-shell (the $\{u_i, -u_i\}$'s can be chosen at random)
- Potentially nice if the overlap formula involves $Q_2(0)\mathbb{D}$
- Factor K_1 signals that a hws is mapped to a descendent

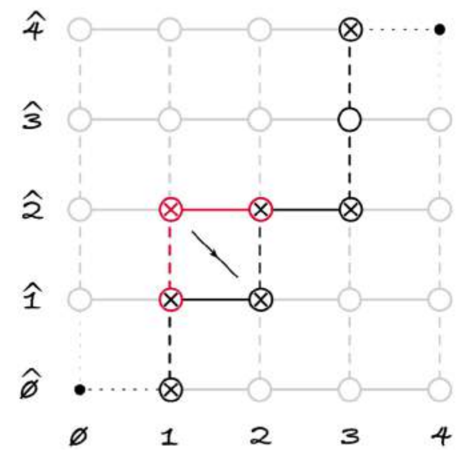
de Leeuw, Ipsen, C.K., Vardinghus, Wilhelm 17

Fermionic dualities in general

- Allow one to move between any two Dynkin diagrams of a super Lie algebra (of type $GL(N|M)$)

- Involve a fermionic node and its neighbours only



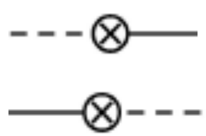


- Changes the nature of neighbouring nodes $\bigotimes \longleftrightarrow \bigcirc$ and the connections ——— $\longleftrightarrow$ - - -

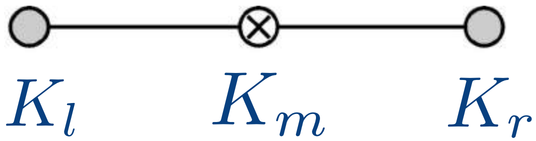
- Dualized node non-momentum carrying $\Longrightarrow$ Dynkin labels unchanged

- Dualized node momentum carrying $\Longrightarrow$ Dynkin labels change

$$\begin{bmatrix} 0 \\ V \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} V \pm 1 \\ -V \\ V \mp 1 \end{bmatrix} \quad \text{for}$$



Dualizing a non-momentum-carrying node



$$M = \begin{bmatrix} \eta_2 & \eta_1 & 0 \\ \eta_1 & 0 & -\eta_1 \\ 0 & -\eta_1 & \eta_3 \end{bmatrix} , \qquad q = \begin{bmatrix} V_l \\ 0 \\ V_r \end{bmatrix} , \qquad \begin{matrix} \eta_1 \in \{-1, +1\} \\ \eta_2 \in \{0, -2\eta_1\} \\ \eta_3 \in \{0, 2\eta_1\} \end{matrix} ,$$

$$K_l, K_r, K_m \text{ all even} \implies \widetilde{K}_m = K_l + K_r - K_m - 1 \text{ odd}$$

$$Q_l^- Q_r^+ - Q_l^+ Q_r^- = i\eta_1 (K_r - K_l) u \, Q_m \widetilde{Q}_m \, ,$$

C.K., Müller, Zarembo '20

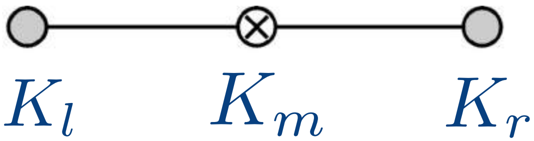
$$\widetilde{\mathbb{D}} = \mathrm{J}\, \mathbb{D} = (-\eta_1)^{K_l} \eta_1^{K_r} \, (\eta_1 K_r - \eta_1 K_l) \, \frac{Q_m(0) \widetilde{Q}_m(0)}{Q_l\left(\frac{i}{2}\right) Q_r\left(\frac{i}{2}\right)} \, \mathbb{D}$$

Found numerically
Analytical proof
in progress

K_l, K_r even, K_m odd

$$\widetilde{\mathbb{D}} = (-\mathrm{J})^{-1} \, \mathbb{D} \, ,$$

Dualizing a momentum-carrying node



$$M = \begin{bmatrix} \eta_2 & \eta_1 & 0 \\ \eta_1 & 0 & -\eta_1 \\ 0 & -\eta_1 & \eta_3 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ V \\ 0 \end{bmatrix}, \qquad \begin{array}{l} \eta_1 \in \{-1, +1\} \\ \eta_2 \in \{0, -2\eta_1\} \\ \eta_3 \in \{0, 2\eta_1\} \end{array}$$

K_l, K_r, K_m, L all even $\implies \widetilde{K}_m = L + K_l + K_r - K_m - 1$ odd

$$\left(u + V\tfrac{i}{2}\right)^L Q_l^- Q_r^+ - \left(u - V\tfrac{i}{2}\right)^L Q_l^+ Q_r^- = i(VL - \eta_1 K_l + \eta_1 K_r)\, u\, Q_m \widetilde{Q}_m,$$

$$\widetilde{\mathbb{D}} = \left(\frac{2i}{V}\right)^L (VL - \eta_1 K_l + \eta_1 K_r)\, \frac{Q_m(0)\widetilde{Q}_m(0)}{Q_l\left(\frac{i}{2}\right) Q_r\left(\frac{i}{2}\right)}\, \mathbb{D}\,,$$

Found numerically
Analytical proof
in progress

NB: K_r odd or K_l odd requires regularization

Dualizing overlap formulas I

K_1	K_2	K_3		K_1	$\widetilde{K}_2$	K_3		K_1	$\widetilde{K}_2$	$\widetilde{K}_3$
E	E	E		E	O	E		E	O	E
E	O	E		E	E	O		E	E	E

$$C = \frac{Q_1(0)Q_2(0)}{Q_1(\frac{i}{2})Q_3(0)Q_3(\frac{i}{2})}\mathbb{D}$$

Gombor & Bajnok '20

$$\longrightarrow \widetilde{C} = \frac{1}{K_1 - K_3}\,\frac{Q_1(0)}{\widetilde{Q}_2(0)Q_3(0)}\widetilde{\mathbb{D}}$$

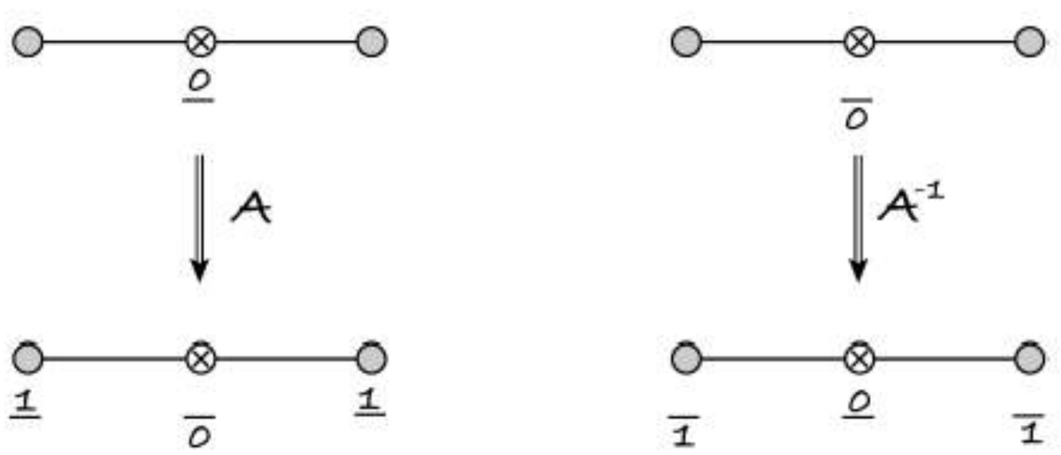
$$\longrightarrow \widetilde{\widetilde{C}} = \frac{\widetilde{K}_2}{K_1 - \widetilde{K}_2 + \tilde{K}_3 + 1}\,\frac{\widetilde{Q}_3(0)\,Q_1(0)}{\widetilde{Q}_2(0)\widetilde{Q}_2(\frac{i}{2})}\widetilde{\widetilde{\mathbb{D}}}$$

C.K., Müller, Zarembo '20

Agreement for both cases

Dualizing overlap formulas II

$$\mathbb{D} \propto \frac{Q_{a-1}\left(\frac{i}{2}\right) Q_{a+1}\left(\frac{i}{2}\right)}{\widetilde{Q}_a(0) Q_a(0)} \widetilde{\mathbb{D}}$$

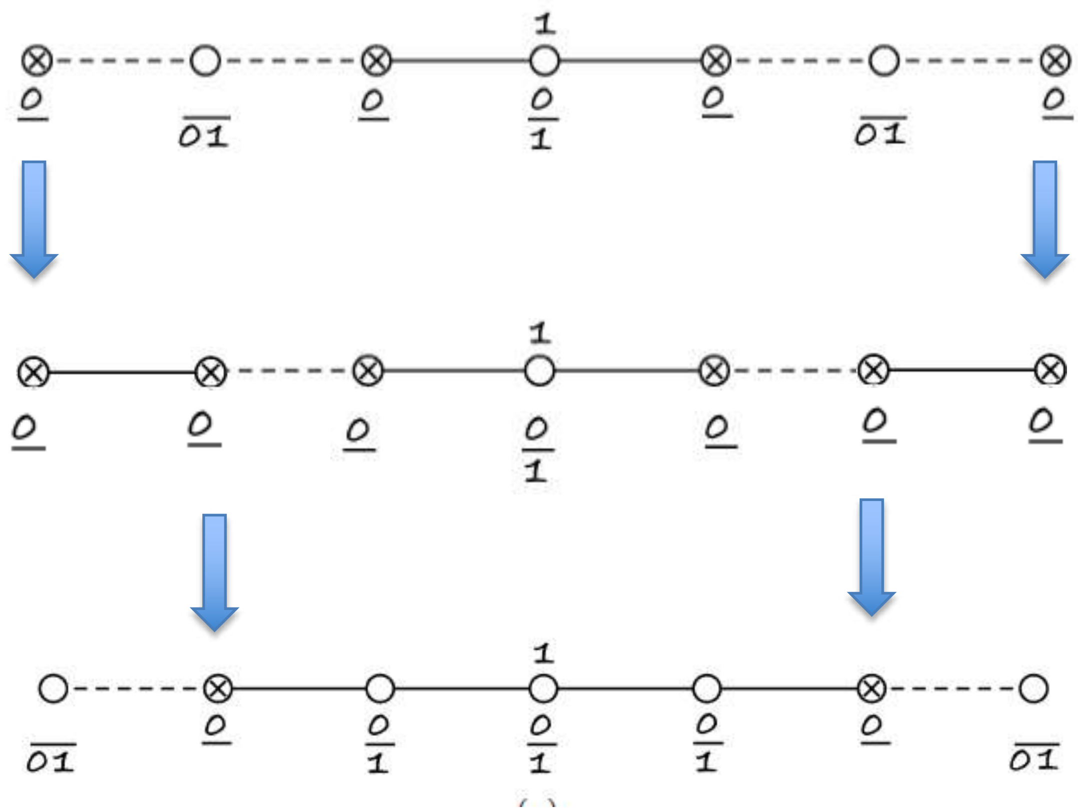


Covariance of overlap formulas very constraining (fully constraining?)

Dualizing overlap formulas III

$PSU(2,2|4)$ overlap formula, alternating grading

Gombor & Bajnok '20

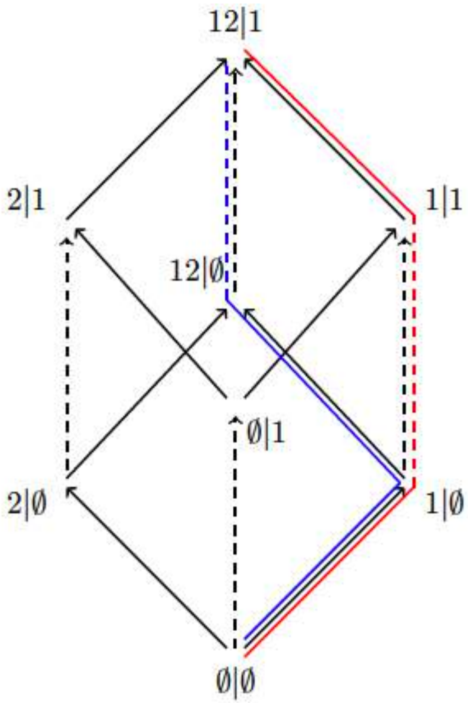


$PSU(2,2|4)$ overlap formula, beauty grading

Agrees with field theory result in $SO(6)$ sector

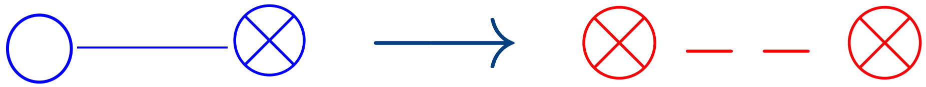
C.K., Müller, Zarembo '20

Bosonic Dualities: Ex: $SU(2|1)$



$Q_{\emptyset|\emptyset} = u^L, \;\; Q_{12|1} = 1$

Fermionic Duality considered so far:



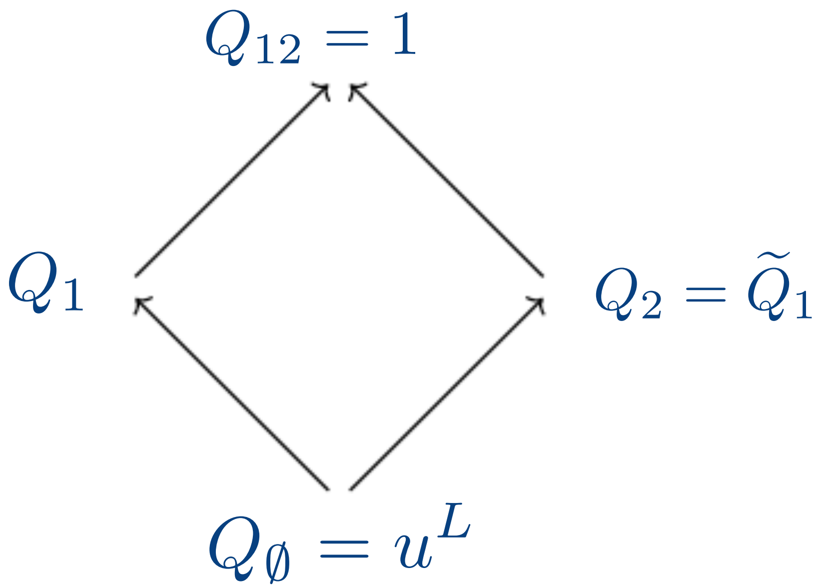
$$Q_{12|\emptyset} Q_{1|1} = Q^+_{12|1} Q^-_{1|\emptyset} - Q^-_{12|1} Q^+_{1|\emptyset} = Q^-_{1|\emptyset} - Q^+_{1|\emptyset}$$

Additional Bosonic Symmetries such as

$$Q^+_{1|\emptyset} Q^-_{2|\emptyset} - Q^-_{1|\emptyset} Q^+_{2|\emptyset} = Q_{\emptyset|\emptyset} Q_{12|\emptyset}$$

How does $\mathbb{D}$ transform under the full set of QQ relations ?

Bosonic Dualities: A warm-up example: $SU(2)$



Bosonic duality eqn.

$$Q_1^+ \widetilde{Q}_1^- - Q_1^- \widetilde{Q}_1^+ = u^L$$

$$\widetilde{K}_1 = L - K_1 + 1$$

Dual roots at $\pm\frac{i}{2}$ call for regularization of $\det G$

After regularization:

$$\widetilde{\mathbb{D}} = -\frac{2\left(2^{L/2-M}(L/2-M)!\right)^4}{(L-2M)!\,(L-2M+1)!}\,\frac{Q(i/2)\widetilde{Q}(0)}{Q(0)\widetilde{Q}(i/2)}\,\mathbb{D}\,,$$

Roots at $0, \pm\frac{i}{2}$ left out in $\widetilde{Q}$

Towards a proof of the transformation formula

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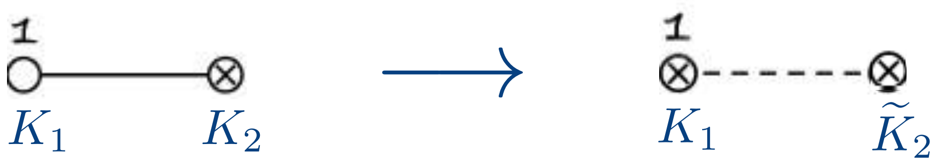
Matrix on block form

$$G = \begin{bmatrix} A & B \\ B^t & E \end{bmatrix} = \begin{bmatrix} 1 & B \\ 0 & E \end{bmatrix} \begin{bmatrix} A - BE^{-1}B^t & 0 \\ E^{-1}B^t & 1 \end{bmatrix},$$

$$\det G = \det E \det(A - BE^{-1}B^t) \equiv \det E \det W$$



$$G^{\pm} = \begin{bmatrix} G^{\pm}_{11} & G^{\pm}_{12} \\ (G^{\pm}_{12})^t & G^{\pm}_{22} \end{bmatrix}, \qquad \widetilde{G}^{+} = \left[\begin{array}{c|cc} \widetilde{G}^{+}_{11} & \widetilde{G}^{+}_{12} & \sqrt{2}\widetilde{G}_{10} \\ \hline (\widetilde{G}^{+}_{12})^t & \widetilde{G}^{+}_{22} & 0 \\ \sqrt{2}\widetilde{G}^{t}_{10} & 0 & \widetilde{G}_{00} \end{array}\right], \qquad \widetilde{G}^{-} = \begin{bmatrix} \widetilde{G}^{-}_{11} & \widetilde{G}^{-}_{12} \\ (\widetilde{G}^{-}_{12})^t & \widetilde{G}^{-}_{22} \end{bmatrix},$$

$G^{+}_{22} = G^{-}_{22} = G_{22},$ diagonal.

$$\begin{aligned} \widetilde{G}^{+}_{11} &= \widetilde{G}^{-}_{11} = \widetilde{G}_{11}, && \text{diagonal,} \\ \widetilde{G}^{+}_{22} &= \widetilde{G}^{-}_{22} = \widetilde{G}_{22}, && \text{diagonal,} \end{aligned}$$

Towards a proof of the transformation formula

Using Schur complements

$$\mathbb{D} = \frac{\det W^+}{\det W^-}, \qquad \widetilde{\mathbb{D}} = \widetilde{G}_{00} \frac{\det \widetilde{W}^+}{\det \widetilde{W}^-},$$

$$\widetilde{G}_{00} = -\sum_{j=1}^{K_1} \frac{1}{u_j^2 + \frac{1}{4}} = \ldots \text{using QQ relations} \ldots = K_1 \frac{Q_2(0)\widetilde{Q}_2(0)}{Q_1(\frac{i}{2})},$$

Still need to show

$$\frac{\det \widetilde{W}^+}{\det \widetilde{W}^-} = \frac{\det W^+}{\det W^-}$$

Idea: Use recursive strategy à la Korepin, Slavnov et al.

Work in progress

Future Directions

- Tranformation of overlap under bosonic dualities (full Hasse diagram)
- Complete the proof of the duality transformation formulas
- Derive the overlap formula on the basis of dualities alone
- Relate the approach to the Quantum Spectral Curve (relevant for higher loops)

Thank you