Line Defects and Renormalization Group Flows

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The subject of line defects has been historically extremely productive. The Kondo defect (which is essentially a line defect in 2d) has led to both the **renormalization group** [Wilson...] and to substantial progress on **integrability** [Andrei, Tsvelick-Wiegmann...]. The topic of this talk is to explore line defects in higher dimensions.

We are already familiar with many constructions of line defects in $d > 2$:

- Wilson/'t Hooft loops.
- Twist (symmetry) defects
- SPT defects
- Worldlines of anyons in 2+1 dimensions
- $\cdots$

This talk will focus on two subjects:

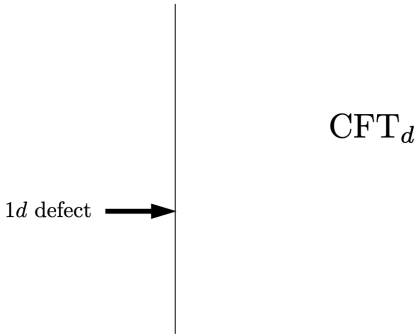
- RG flows on line defects
- The limit of “heavy” line defects

We will discuss various applications of both subjects.

We will consider lines in a d-dimensional CFT. Consider a straight line. It can be conformal or non-conformal. A conformal line preserves

$$SL(2,R) \times SO(d-1)$$

(we assume the line has no transverse spin). It describes a critical point-like impurity in space.

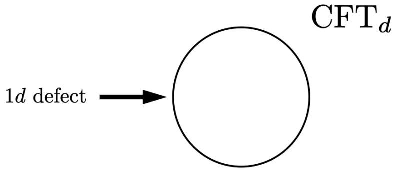

CFT_d
$1d$ defect

An interesting observable for such a line is its defect entropy (not the same as entanglement entropy for $d > 2$). We make the line into a circle and compute the expectation value of the circle.

$$s = \left(1 - R\frac{\partial}{\partial R}\right)\log\langle L\rangle \equiv \log g \ .$$

From conformal invariance it would seem that $\langle L\rangle$ is R-independent but in fact it is possible that there is linear in R divergence in $\log\langle L\rangle$ which is just mass-renormalization of the impurity. This cancels out from s.

Therefore s is a scheme-independent intrinsic observable. At the fixed point of the line defect the value of s is also called g. For line defects in a 2+1 dimensional topological theory, g is called the "quantum dimension." When the bulk is a topological theory, it always satisfies $g \geq 1$. $g = 1$ for Abelian anyons (one-form symmetry lines).



More generally, we know that $g \geq 0$ but it is not necessarily true that $g \geq 1$.
In the general case, line defects are called topological if you can wiggle the worldline infinitesimally around without changing the partition function. The response to wiggling the line is called the displacement operator $D^i(t)$ and a line is called topological if

$$D^i(t) = 0 \ .$$

If a topological line is invertible then clearly $g = 1$.

Defect operators are classified by their $SL(2, R) \times SO(d-1)$ quantum numbers. For instance for the trivial line defect (the completely transparent line), or for any invertible topological line, the defect operators coincide with the bulk operators. More generally, the space of defect operators has nothing to do with the bulk operators.

In the event that relevant defect operators exist, we can add it to the action $M_0^{1-\Delta}\int dt O(t)$. M_0 becomes the physical scale of the flow. We can again study the circular line and consider $s=\left(1-R\frac{\partial}{\partial R}\right)\log\langle L\rangle$ which now becomes a nontrivial function

$$s = s(M_0 R)$$

We have

$$s(M_0R)\to\begin{cases}\log g_{UV} \text{ as } R\to 0\\ \log g_{IR} \text{ as } R\to\infty\end{cases}$$

For conformal defects, the energy momentum tensor of the defect vanishes $T_D = 0$ since it has only one index and it has to be traceless. Hence, a conformal line defect cannot support localized energy [e.g. Herzog-Huang].
For non-conformal defects, T_D measures the energy density on the defect and it is a nonzero operator.

The main result is the following identity

$$M_0 \frac{\partial s}{\partial M_0} = -R^2 \int d\phi_1 d\phi_2 \langle T_D(\phi_1) T_D(\phi_2)\rangle_c \left(1 - \cos(\phi_1 - \phi_2)\right) \ .$$

Since $\langle T_D(\phi_1) T_D(\phi_2)\rangle_c \geq 0$ at separated points and since $(1 - \cos(\phi_1 - \phi_2)) \geq 0$ we have that

$$M_0 \frac{\partial s}{\partial M_0} \leq 0 \ ,$$

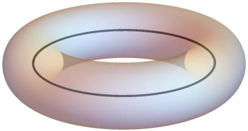
and therefore also $g_{UV} \geq g_{IR}$.

The main idea of the derivation is to promote the massive perturbation to a time dependent one

$$M_0^{1-\Delta}\int dt O(t) \longrightarrow M_0^{1-\Delta}\int dt e^{(1-\Delta)\Phi(t)} O(t) \ .$$

$\Phi(t)$ is usually called the dilaton though it has nothing to do with the string theory dilaton. $\Phi(t)$ is a classical field (background field).

The Ward identities of the ambient d-dimensional CFT impose some relations between theories with different profiles $\Phi(t)$. The main relation we will use comes from integrating over a surface wrapping the circle defect



$$Q_\xi = \int_{\text{torus}} d^{d-1}\Sigma^\mu T_{\mu\nu}\xi^\nu \ .$$

On the one hand it must be that

$$Q_\xi = 0$$

from the conformal invariance of the bulk vacuum. On the other hand we shrink the torus around the line and create a new line defect.

The main computation to do is therefore to find which new line defect is created by shrinking the torus around the line defect. The answer is that you create a new defect with

$$\tilde{\Phi} = \Phi + \alpha \left(\dot{\xi}_D + \xi_D \dot{\Phi} \right) \ ,$$

where ξ_D is a conformal Killing vector restricted to the defect and α is an infinitesimal parameter. By choosing $\Phi(t)$ and ξ_D appropriately this can be used to derive an interesting constraint for the correlation functions on circular defects:

$$R \int d\phi \langle T_D(\phi) \rangle = R^2 \int d\phi_1 d\phi_2 \langle T_D(\phi_1) T_D(\phi_2) \rangle_c \cos(\phi_1 - \phi_2) \ .$$

This constraint can be manipulated a little and turned into the sum rule that leads to $M_0 \frac{\partial s}{\partial M_0} \leq 0$.
This generalizes the familiar results of [Affleck-Ludwig, Friedan-Konechny] to line defects/impurities in higher dimensions.

Note that it follows that g is independent of exactly marginal defect couplings.

A very simple sanity check of this result is the flow [Polchinski-Sully]

$$\text{Wilson Loop} \longrightarrow \text{Maldacena} - \text{Wilson Loop}$$

We define the defect

$$W^{\zeta} = TrPe^{\int dt (iA_{\mu}\dot{x}^{\mu} + \zeta \Phi_m(x)\theta^m |\dot{x}|)}$$

ζ is a parameter which flows, $\theta^2 = 1$, and at weak 't Hooft coupling and large N we find

$$\beta_{\zeta} = -\frac{\lambda}{8\pi^2}\zeta(1-\zeta^2) + \mathcal{O}(\lambda^2)$$

we checked by a computation of $\langle T_D(\phi_1) T_D(\phi_2)\rangle_c$ that the sum rule is satisfied and one can also find the defect entropy function explicitly and verify that it is monotonically decreasing. We used many results of [Beccaria-Giombi-Tseytlin].

An even simpler model is to start from the trivial defect in the free theory and since the spectrum of defect operators is the same as the spectrum of bulk operators restricted to the defect we can consider

$$\int d^d x \frac{1}{2}(\partial\phi)^2 + M_0^{\frac{4-d}{2}} \int_{defect} dt \ \phi$$

For a circular defect of radius R we can compute the partition function analytically and find the defect entropy

$$s = \pi(d-3)\frac{2^{1-d}\pi^{\frac{3}{2}-d}\Gamma\left(\frac{3}{2}-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(2-\frac{d}{2}\right)}(RM_0)^{4-d}$$

The defect entropy decreases monotonically as expected, the identity relating the defect entropy flow to $\langle T_D T_D \rangle_c$ is satisfied

$$s(M_0 R) \to \begin{cases} 0 \text{ as } R \to 0 \\ -\infty \text{ as } R \to \infty \end{cases}$$

In the ultraviolet we have the trivial defect and in the infrared we have $s \to -\infty$, so the flow does not quite end at an IR DCFT. It is tempting to conjecture that such a non-terminating flow can only happen in theories with a bulk moduli space of vacua.

In $d = 4$ the line defect is conformal for all ζ and it was studied in [Kapustin]. It has $g = 1$ but it is not a trivial defect and other observables do depend on ζ.

The problem of line defects is quite interesting and nontrivial even when the bulk theory is completely free. Let us consider 3 free fields ϕ^a in d dimensions

$$\int d^d x \frac{1}{2}(\partial\phi^a)^2$$

and a defect on a line, which is an impurity in the spin S representation of the bulk global $SO(3)$ symmetry [Sengupta, Sachdev-Buragohain-Vojta, Liu-Shapourian-Vishwanath-Metlitski]:

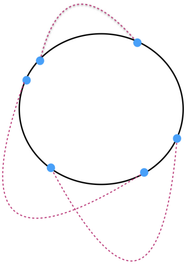
$$P e^{i\zeta \int dt \phi^a T^a}$$

with T^a the $(2S+1)\times(2S+1)$ matrices of the $su(2)$ Lie algebra.

This physically describes a spin S impurity in a ferromagnet, although for simplicity we are not at the true interacting bulk fixed point. $Pe^{i\zeta\int dt\phi^a T^a}$ defines an $SU(2)$ matrix transforming under the bulk symmetry by $Pe^{i\zeta\int dt\phi^a T^a} \to gPe^{i\zeta\int dt\phi^a T^a}g^{-1}$. The case of the impurity being truly point-like and interacting in an $SO(3)$ invariant fashion with the bulk thus corresponds to

$$Tr_S\left(Pe^{i\zeta\int dt\phi^a T^a}\right)$$

Note that this looks like a Wilson line but there is no gauge symmetry in this problem. And even though the bulk is free, computing the defect entropy, scaling dimensions of defect operators and so on is difficult.



The parameter ζ is relevant for $d < 4$ and marginally irrelevant in $d = 4$ for all S. (The free defect is stable in $d = 4$.)

$$\log g = \log(2S+1) - \frac{(4-d)}{8} S^2 \zeta^2 + \frac{\zeta^4 S^2}{32\pi^2} + \cdots$$

$$\beta_\zeta = \frac{1}{2}(d-4)\zeta + \frac{1}{4\pi^2}\zeta^3 + \cdots$$

The UV DCFT, $\zeta = 0$, is not the trivial DCFT; it is a decoupled impurity with $2S+1$ states and $g = 2S+1$. It is completely transparent and topological but it is not the trivial line, as can be seen from the space of defect operators in the UV theory.

$$\log g = \log(2S+1) - \frac{(4-d)}{8} S^2\zeta^2 + \frac{\zeta^4 S^2}{32\pi^2} + \cdots$$

$$\beta_\zeta = \frac{1}{2}(d-4)\zeta + \frac{1}{4\pi^2}\zeta^3 + \cdots$$

We see a weakly coupled fixed point with $\zeta^2 = 2\pi^2(4-d)$ and $\log g_{IR} = \log(2S+1) - \frac{1}{8}S^2\pi^2(4-d)^2 < \log g_{UV}$.
This fixed point is stable for $SO(3)$ preserving perturbations.

In the analysis above we have assumed that S is fixed while $4-d$ and $\zeta^2 \sim 4-d$ are the smallest parameters.

A possible higher order term in $\log g$ is $\zeta^6 S^4$ and in β_ζ it is $\zeta^5 S^2$. This is just because $SU(2)$ generators scale like $T^a \sim S$. Therefore, naively, for $S \sim (4-d)^{-1/2}$ the expansion breaks down.

It turns out that there are vast cancelations and the expansion reorganizes in terms of $\zeta^2 S$ so it breaks down for $S \sim (4-d)^{-1}$. Furthermore, the problem turns out to be solvable for large S and arbitrary fixed $\zeta^2 S \equiv \alpha$.

Let us describe the main idea of how one would approach such a limit of "heavy" defects. It seems pretty difficult starting from the expression

$$Tr_S \left(P e^{\zeta \int dt \phi^a T^a} \right) \ .$$

An alternative viewpoint is to introduce first order variables on the line defect $z = (z_1, z_2)$ and represent the Wilson line with the action

$$S_{defect} = \left(S + \frac{1}{2} \right) \int dt \left(\bar{z}\dot{z} + \frac{1}{2} \zeta z \sigma^a \bar{z} \phi^a \right) \ .$$

This is subject to the constraint $z\bar{z} = 2$ and the gauge invariance $z(t) \to z(t) e^{i\alpha(t)}$, so the configuration space is really just the two-sphere.

$$S_{defect} = \left(S + \frac{1}{2}\right) \int dt \left(\bar{z}\dot{z} + \frac{1}{2}\zeta z\sigma^a \bar{z}\phi^a\right) \ .$$

This representation of the Wilson line as a first order kinetic term on S^2 is essentially the co-adjoint orbit method. The advantage of this approach is that $\hbar \sim \frac{1}{S}$, so it allows a systematic expansion in S^{-1}. To see it in more detail rescale $\phi \to \phi(S + 1/2)^{1/2}$, the whole action including the bulk becomes

$$S_{defect} = \left(S + \frac{1}{2}\right) \int d^d x \frac{1}{2}(\partial\phi^a)^2$$

$$+ \left(S + \frac{1}{2}\right) \int dt \left(\bar{z}\dot{z} + \frac{1}{2}\zeta \left(S + 1/2\right)^{1/2} z\sigma^a \bar{z}\phi^a\right) \ .$$

Therefore at large S and fixed $\zeta^2 S$ we have a new emergent classical limit and we can quantize around its saddle points.

Such a semi-classical approach towards large quantum numbers was studied in [Badel-Cuomo-Monin-Rattazzi] in the context of bulk scaling dimensions in the $O(2)$ model. A similar semi-classical approach to large quantum numbers in supersymmetric theories was studied by many other authors.
Here this idea is applied for a line defect in the (free) $O(3)$ model.

This allows to perform a resummation of various terms in the beta function and $\log g$ and many other observables. Results from the classical limit can be compared with explicit diagrammatic computations in some cases.

Example: the g function in this double scaling limit and for $d = 4$ is

$$\log g = log(2S+1) + \frac{1}{2}\log\left(1 + \frac{\zeta^4 S^2}{16\pi^2}\right) + \mathcal{O}(S^{-1}) \ ,$$

expanding it out, the first nontrivial term agrees with an explicit laborious perturbative computation that we have quoted previously. We also found a new fixed point in this limit, that we are presently studying. One can verify the g theorem for flows in this limit.

This technology of the large S expansion can be used for the interacting Wilson Fisher model as well.

This technology of the large S expansion also transfers almost verbatim to Wilson lines in large representations. We have so far used these tools to study supersymmetric and nonsupersymmetric Wilson loops in $SU(2)$ $\mathcal{N} = 4$ SYM theory.

It is encouraging that one can indeed reproduce detailed predictions of localization through this emergent classical limit.

- The higher dimensional g theorem challenges the connection between monotonicity theorems and entanglement entropy. See [Casini – Salazar-Landea – Torroba] for the $d = 2$ case.
- Is there is a nontrivial lower bound on g in theories with no bulk moduli? Some $d = 2$ bootstrap work on this can be found in [Friedan – Konechny – Schmidt-Colinet]
- Being that g is defined by making the defect into a circle in space-time, it is challenging to think about VEV flows. VEV flows are when the defect is not deformed by an operator, but rather, some defect operator has a nontrivial VEV.
- It would be interesting to combine the limit of heavy defects with bulk large N and also make contact with experimental results on magnetic impurities.
- Upon integrating out the bulk, we can think about the defect as a non local QM. Sometimes it looks like familiar disordered systems from condensed matter. It would be nice to explore this further.

- It would be nice to make contact with AdS/CFT. Can our sum rule for the defect entropy flow be obtained? Is there a useful description of Wilson loops in a very large representation? Connection to the integrability techniques for line defects in $\mathcal{N} = 4$ in the planar limit [Giombi-Komatsu...]?
- It is well understood how to apply the conformal bootstrap philosophy for line defects (or more general defects) [Liendo-Rastelli-van Rees], [Billo-Goncalves-Lauria-Meineri], [Lauria-Liendo-van Rees -Zhao]... It would be nice to explore the fixed points we have found for the magnetic impurity and compare notes.

Thank You!