From Gaudin Integrable Models to *d*-dimensional Multipoint Conformal Blocks

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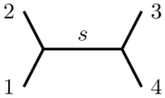
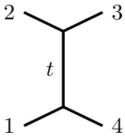
based on I.B, S. Lacroix, J. A. Mann, L. Quintavalle and V. Schomerus, [2009.11882]

Conformal bootstrap

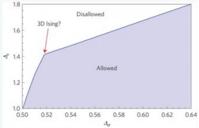
Conformal bootstrap = constrain CFTs using channel duality
Usually, crossing symmetry of four-point functions

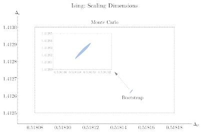
$$\sum_{s\in S} C_{12s}C_{s34}$$


$$= \sum_{t\in S} C_{23t}C_{t41}$$


In practice, crossing is studied for a few simplest operators

Remarkable numerical results



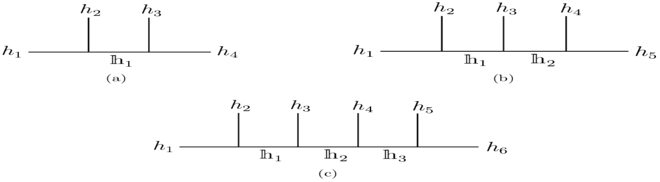


Also successful analytic studies → lightcone bootstrap

Higher-point correlation functions

All four-point functions: sufficient, but not efficient

Idea: study higher-point correlators of simple operators



Difficulty: Many conformal invariants

$$\# = \frac{1}{2}m(m-3) + d(N-m), \quad m = \min(N, d+2)$$

so not clear how to compute conformal blocks. This talk:

Higher-point blocks = eigenfunctions of Gaudin Hamiltonians!

In need of Hamiltonians

Origin: N-point functions = invariants in the N-fold tensor product of principal series reps [Isachenkov, Schomerus, Sobko]

$$G_N(x_i) \in (\pi_1 \otimes \ldots \otimes \pi_N)^G .$$

We need operators that commute with the diagonal action of G.

Gaudin model: Lie algebra $\mathfrak{g} = \mathfrak{so}(d+1,1)$, numbers $z_1, \ldots, z_N \in \mathbb{C}$

Observables: $\mathcal{A} = U(\mathfrak{g})^{\otimes N} = \langle X_a^{(i)} \rangle \quad \rightarrow \quad \pi_1 \otimes \ldots \otimes \pi_N$

Lax matrix and Gaudin Hamiltonians (GHs)

$$\mathcal{L}_a(z) = \sum_{i=1}^{N} \frac{X_a^{(i)}}{z - z_i}, \quad \mathcal{H}_p(z) = \frac{1}{p} \tau^{a_1 \ldots a_p} \mathcal{L}_{a_1}(z) \ldots \mathcal{L}_{a_p}(z) + \ldots$$

Commute between themselves and also with the diagonal action

$$[\mathcal{H}_p(z), \mathcal{H}_q(w)] = 0, \quad [\sum_{i=1}^{N} X_a^{(i)}, \mathcal{H}_p(z)] = 0 .$$

Example: Five-point functions

Question: Do GHs include Dolan-Osborn operators?

$$\mathcal{C}_p^{(ij)} = \frac{1}{p}\kappa_p^{a_1\ldots a_p}(X_{a_1}^{(i)} + X_{a_1}^{(j)})\ldots(X_{a_p}^{(i)} + X_{a_p}^{(j)}) \equiv \frac{1}{p}\kappa_p^{ab}X_{a_1}^{(ij)}\ldots X_{a_p}^{(ij)}$$

Yes! Fix the parameters

$$z_1 = 0, \quad z_2 = \omega, \quad z_3 = \omega^2, \quad z_4 = 1, \quad z_5 = \infty,$$

take the limit $\omega \to 0$ and rescale $\tilde{\mathcal{H}}_p(z) = \omega^p \mathcal{H}_p(\omega z)$

$$\mathcal{C}_p^{(12)} = \lim_{z \longrightarrow 0} z^p \tilde{\mathcal{H}}_p(z^p), \quad \mathcal{C}_p^{(45)} = \lim_{z \longrightarrow \infty} z^p \tilde{\mathcal{H}}_p(z^p)\ .$$

Four such operators and the vertex operator

$$\mathcal{V}_4 = \frac{\tilde{\mathcal{H}}_4(1/2)}{16} + \ldots = \kappa_4^{a_1\ldots a_4}(X_{a_1}^{(12)} - X_{a_1}^{(3)})\ldots(X_{a_4}^{(12)} - X_{a_4}^{(3)})$$

Corresponds to the choice of tensor structures at the vertex of the OPE diagram - characterised by the equation $Ht_n = \tau_n t_n$

Summary and outlook

Result:

1. Multipoint conformal blocks = Gaudin eigenfunctions
2. Distinguished basis of tensor structures at the vertex
3. Today: $N = 5$ but can extend to any N

Goal: Solution theory

1. Double lightcone limit $x_{i,i+1}^2 = 0 \to$ eigenvalue eqn for H
2. Lightcone limit $x_{12}^2 = x_{45}^2 = 0 \to 3$ variable problem
3. Points in general position

First application: Lightcone bootstrap for five-point functions

There is a lot to explore: relations to Wilson loops, defects, the role of Appell functions... can't wait to get back to it!

The End