# Integrability for Feynman Integrals

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*based on*
*Phys.Rev.D 101 (2020) 6, arXiv:1912.05561*
with Dennis Müller, Hagen Münkler

*and Phys.Rev.Lett. 125 (2020) 9, 091602, arXiv:2005.01735*
with Julian Miczajka, Dennis Müller, Hagen Münkler

*and arXiv:2008.11739*
with Julian Miczajka

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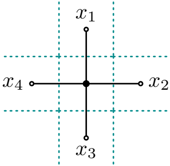
# Motivation

- QFT computations are hard
- Toy model: Planar $\mathcal{N} = 4$ SYM theory
- Highly constraining symmetries (Yangian)
- What does this teach us about generic building blocks of QFT?

Feynman's Van

# Example: Massless Box Integral

Consider Euclidean cross (or box) integral:




$$= \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} = \int \frac{\mathrm{d}^4 \ell}{\ell^2 (\ell + p_1)^2 (\ell + p_1 + p_2)^2 (\ell - p_4)^2}$$

$x$-variables interpreted as coordinates or dual momenta $p_j^\mu = x_j^\mu - x_{j+1}^\mu$

(Dual) conformal symmetry implies

$$I_4 = \frac{1}{x_{13}^2 x_{24}^2} \phi(z, \bar{z})$$

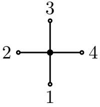
with

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad (1 - z)(1 - \bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

# Box Integral from Yangian Symmetry

Yangian symmetry implies differential equations:

$$[D_j(z) - D_j(\bar{z})]\phi(z,\bar{z}) = 0, \qquad j = 1, 2,$$




with $\begin{cases} D_1(z) = z(z-1)^2\partial_z^2 + (3z-1)(z-1)\partial_z + z, \\ D_2(z) = z^2(z-1)\partial_z^2 + (3z-2)z\partial_z + z. \end{cases}$

Find four solutions $f_j(z,\bar{z})/(z-\bar{z})$ by elementary methods:

$$\begin{aligned} f_1 &= 1, \\ f_2 &= \log(\bar{z}) - \log(z), \\ f_3 &= \log(1-\bar{z}) - \log(1-z), \\ f_4 &= 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log\tfrac{1-z}{1-\bar{z}}\log(\bar{z}z). \end{aligned}$$

Permutation symmetry singles out Bloch-Wigner dilogarithm. [Usyukina Davydychev '93]

⇒ Box integral fixed by symmetries! [FL, Müller Münkler 2019]

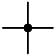
# Outline

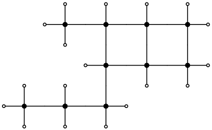
- Review: Integrability of massless fishnet integrals
- Bootstrap for massless integrals
- Yangian symmetry for massive integrals
- Massive momentum conformal symmetry
- Massive fishnet theory

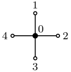
# Yangian Symmetry without Masses

# Massless Fishnet Feynman Graphs

Feynman graphs made from scalar four-point vertices in $4\mathrm{D}$ ($x^2 = x^\mu x_\mu$):

- Vertex  : $\int \mathrm{d}^4 x$
- Propagator $j$  $k$ : $\frac{1}{x_{jk}^2} \equiv \frac{1}{(x_j - x_k)^2}$

e.g. 

Mostly unsolved, only cross known [Ussyukina '93, Davydychev]: 

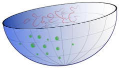
$= \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$

**Remarkable properties**:

- Dual conformal symmetry
- UV & IR finite integrals
- Represent integrable statistical vertex model [Zamolodchikov 1980]
- Related to AdS/CFT integrability ...

# Bi-Scalar Limit of Twisted $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ **SYM Theory:** 4d gauge theory with Lagrangian $\mathcal{L}_{\mathcal{N}=4}$ for scalars, fermions and gluons in adjoint of gauge group $\mathrm{SU}(N_\mathsf{c})$.



- AdS/CFT: Dual to strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$
- Integrable in the planar limit: $N_\mathsf{c} \to \infty$ with fixed $g^2 = g^2_{\mathsf{YM}} N_\mathsf{c}$.

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**Three-parameter $\gamma$-Deformation**: [Leigh '95 Strassler] [Lunin '05 Maldacena] [Frolov '05]

Introduce non-commutative field products in the $\mathcal{N} = 4$ Lagrangian:

$$X\,Y \mapsto X \star Y := e^{-\frac{i}{2}\gamma_j \epsilon^{jkl} Q^X_k Q^Y_l} X\,Y, \qquad X, Y \in \{\text{fields}\}$$

R-symmetry charges

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**Double-scaling limit:** [Gürdoğan Kazakov 2015]

Field content reduced by tuned limits of parameters ($\to$ no gluons!):

New couplings

$$g \to 0, \qquad \gamma_{1,2,3} \to i\infty, \qquad \xi_j = g e^{-\frac{i}{2}\gamma_j} = \text{fix.}$$

**Relation to fishnet Feynman graphs?**

# Feynman Graphs as Correlators

| $\underline{\mathcal{N}=4\text{ SYM}}$ | | $\underline{\gamma\text{-Deformation}}$ | | $\underline{\text{Fishnets}}$ |
|---|---|---|---|---|
| $\mathcal{L}_{\mathcal{N}=4}$ | $\xrightarrow{XY\to X\star Y}$ | $\mathcal{L}^{\gamma}_{\mathcal{N}=4}$ | $\xrightarrow[\xi_3=ge^{-i\frac{\gamma_3}{2}}\text{ fix}]{g\to 0,\ \gamma_3\to i\infty,\ \xi_{1,2}=0}$ | $\mathcal{L}_{\text{bi}}$ |

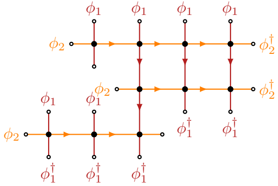
Resulting chiral, non-unitary, bi-scalar theory: [Gürdoğan Kazakov 2015]

$$\mathcal{L}_{\text{bi}} = N_{\text{c}}\,\text{Tr}\left(\partial^\mu\phi_1^\dagger\partial_\mu\phi_1 + \partial^\mu\phi_2^\dagger\partial_\mu\phi_2 + \xi^2\,\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2\right).$$

Consider correlators

$$K = \langle\text{tr}\left[\chi_1(x_1)\chi_2(x_2)\dots\chi_n(x_n)\right]\rangle, \qquad \chi_j\in\{\phi_1,\phi_2,\phi_1^\dagger,\phi_2^\dagger\}.$$

Each correlator given by single fishnet Feynman graph, e.g.




Where is the integrability of planar $\mathcal{N}=4$ SYM theory?

# Integrability in $\mathcal{N}=4$ SYM Theory: Yangian

- The Yangian is an infinite dimensional extension of a Lie algebra $\mathfrak{g}$.
- It underlies rational quantum integrable models (e.g. $\text{AdS}_5/\text{CFT}_4$).

Yangian algebra $Y[\mathfrak{g}]$ (first realization): [Drinfel'd 1985]

Level 0 : $\mathrm{J}^a = \sum_{k=1}^{n} \mathrm{J}_k^a \in \mathfrak{g}$, $\quad [\mathrm{J}^a, \mathrm{J}^b] = f^{ab}{}_c \mathrm{J}^c$

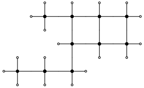
Level 1 : $\widehat{\mathrm{J}}^a = f^a{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}_j^c \, \mathrm{J}_k^b + \sum_{j=1}^{n} s_j \mathrm{J}_j^a$, $\quad [\mathrm{J}^a, \widehat{\mathrm{J}}^b] = f^{ab}{}_c \widehat{\mathrm{J}}^c$

Serre: $[\widehat{\mathrm{J}}_a, [\widehat{\mathrm{J}}_b, \mathrm{J}_c]] - [\mathrm{J}_a, [\widehat{\mathrm{J}}_b, \widehat{\mathrm{J}}_c]] = \mathcal{O}(\mathrm{J}^3)$.

- In $\mathcal{N}=4$ SYM theory we have $\mathfrak{g} = \mathfrak{psu}(2,2|4)$.

# Conformal Yangian for Fishnets

Fishnet integrals inherit a conformal Yangian symmetry from $\mathcal{N}=4$ SYM theory with $\mathfrak{g}=\mathfrak{so}(1,5)$:

$\widehat{\mathrm{J}}^a$  $= 0.$

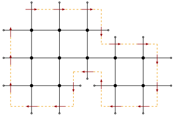
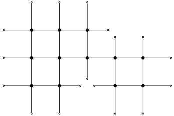
Can be proven using monodromy [Chicherin, Kazakov, FL, Müller, Zhong 2017]

$$T(u) \simeq \mathbb{1} + \tfrac{1}{u}\mathrm{J} + \tfrac{1}{u^2}\widehat{\mathrm{J}} + \dots$$

built from Lax operators:

$$T(\vec{u}) = L_n(u_n;\Delta_n)L_{n-1}(u_{n-1};\Delta_{n-1})\dots L_1(u_1;\Delta_1).$$

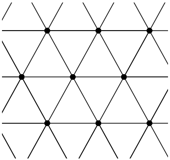
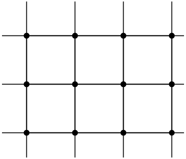
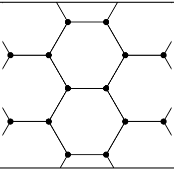
Yangian invariance corresponds to eigenvalue equation:

 $= f(u)$ 

Only square fishnets allowed?

# Regular Tilings of the Plane

[Chicherin, Kazakov FL, Müller Zhong 2017]

| Dimension | $D = 3$ | $D = 4$ | $D = 6$ |
|---|---|---|---|
| Propagator | $\lvert x_{ij} \rvert^{-1}$ | $\lvert x_{ij} \rvert^{-2}$ | $\lvert x_{ij} \rvert^{-4}$ |
| Scalar Fishnet |  |  |  |

Generalizes to $D$-dimensional integrals with generic propagator powers.

[FL, Müller Münkler 2020] [FL, Miczajka Müller,Münkler 2020]

What do Yangian constraints imply?

# Yangian Bootstrap

# Yangian Level Zero

**Conformal Lie Algebra Symmetry**:

$$\mathrm{J}^a \begin{cases} \mathrm{D} & = -ix_\mu \partial^\mu - i\Delta, \\ \mathrm{L}_{\mu\nu} & = ix_\mu \partial_\nu - ix_\nu \partial_\mu, \\ \mathrm{P}_\mu & = -i\partial_\mu, \\ \mathrm{K}_\mu & = ix^2 \partial_\mu - 2ix_\mu x^\nu \partial_\nu - 2i\Delta x_\mu. \end{cases}$$

If propagator powers $a_k$ at each vertex satisfy conformal condition

$$\sum_{k \in \text{vertex}} a_k = D$$

integrals $I_n$ in $D$ dimensions obey

$$\mathrm{J}^a I_n = 0.$$

$\Rightarrow$ dependence on $N$ ( $\simeq n(n-3)/2$) cross ratios:

$$I_n = V_n \, \phi(u_1, \ldots, u_N).$$

# Yangian Level One

Level zero plus one level-one generator sufficient for full Yangian symmetry, e.g. level-one momentum generator:

$$\widehat{\mathrm{P}}^{\mu} = \tfrac{i}{2} \sum_{j,k=1}^{n} \mathsf{sign}(k-j)\big(\mathrm{P}_j^{\mu}\mathrm{D}_k + \mathrm{P}_{j\nu}\mathrm{L}_k^{\nu\mu}\big) + \sum_{j=1}^{n} s_j \mathrm{P}_j^{\mu}.$$

Yangian invariance of $I_n = V_n\, \phi$:

$$0 = \widehat{\mathrm{P}}^{\mu} I_n = V_n \sum_{j<k=1}^{n} \frac{x_{jk}^{\mu}}{x_{jk}^{2}}\, \mathrm{PDE}_{jk}\phi$$

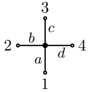
leads to PDEs in the cross ratios:

$$\mathrm{PDE}_{jk}\,\phi = 0, \qquad 1 \le j < k \le n.$$

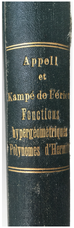
# Example I: Cross (box) in $D$ Dimensions

Deformed cross integral:

$$\int \frac{\mathrm{d}^D x_0}{x_{10}^{2a} x_{20}^{2b} x_{30}^{2c} x_{40}^{2d}} =$$




$$a+b+c+d=D$$




Yangian constraints in $u = x_{12}^2 x_{34}^2 / x_{13}^2 x_{24}^2$ and $v = x_{14}^2 x_{23}^2 / x_{13}^2 x_{24}^2$:

$$0=\Big[\alpha\beta+(\alpha+\beta+1)u\partial_u+\big((\alpha+\beta+1)v-\gamma'\big)\partial_v+u^2\partial_u^2+(v-1)v\partial_v^2+2vu\partial_u\partial_v\Big]\phi(u,v),$$

$$0=\Big[\alpha\beta+(\alpha+\beta+1)v\partial_v+((\alpha+\beta+1)u-\gamma)\partial_u+v^2\partial_v^2+(u-1)u\partial_u^2+2vu\partial_v\partial_u\Big]\phi(u,v)\,.$$

with parameters

$$\alpha=b, \qquad \beta=\frac{D}{2}-d, \qquad \gamma=+\frac{D}{2}-c-d+1, \qquad \gamma'=-\frac{D}{2}+b+c+1.$$

Defining PDEs for Appell hypergeometric function $F_4$:

$$F_4\left[{\alpha,\beta \atop \gamma,\gamma'};u,v\right] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n(1)_m(1)_n}\, u^m v^n\,,$$

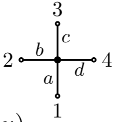
Pochhammer symbol: $(\lambda)_k = \Gamma(\lambda+k)/\Gamma(\lambda)$.

# Permutation Symmetries

Algorithm yields four-dimensional solution space:

$$g_1 = F_4\left[{\alpha,\beta \atop \gamma,\gamma'};u,v\right],$$
$$g_2 = u^{1-\gamma} F_4\left[{\alpha+1-\gamma,\beta+1-\gamma \atop 2-\gamma,\gamma'};u,v\right],$$
$$g_3 = v^{1-\gamma'} F_4\left[{\alpha+1-\gamma',\beta+1-\gamma' \atop \gamma,2-\gamma'};u,v\right],$$
$$g_4 = u^{1-\gamma} v^{1-\gamma'} F_4\left[{\alpha+2-\gamma-\gamma',\beta+2-\gamma-\gamma' \atop 2-\gamma,2-\gamma'};u,v\right].$$

Fix linear coefficients from permutation symmetries




$$\phi(u,v) = c_1\, g_1(u,v) + c_2\, g_2(u,v) + c_3\, g_3(u,v) + c_4\, g_4(u,v)$$

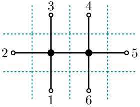
- Limit $a,b,c,d \to 1$ reproduces Bloch–Wigner Function.
- Three-point limit agrees with result of [Boos '90 Davydychev].

$\Rightarrow$ $D$-dimensional Box integral fixed by symmetries! [FL,Müller Münkler 2019]

# Example II: Massless Double Box

Yangian PDEs

$$D_I\,\phi(u_1,\dots,u_9)=0, \qquad I\in\{A,B,C,D,E,F\},$$




with differential operators

$$D_A=-\theta_6^2+u_6(D_{168}+1)D_{365}+u_5u_6(D_{168}+1)(D_{2534}+1)-u_6u_8D_{365}D_{1928}$$
$$+u_4u_5u_6D_{142}(D_{168}+1)+u_6u_8u_9D_{365}D_{392}-u_5u_6u_7u_8D_{1928}(D_{2534}+1),$$
$$D_B=\theta_8(D_{168}+1)-u_8D_{1928}D_{5867}-u_7u_8D_{475}D_{1928}+u_8u_9D_{392}D_{5867},$$
$$D_C=(\theta_1-\theta_9)D_{1928}-u_1D_{142}(D_{168}+1)+u_9D_{392}(D_{798}+1),$$
$$D_D=-\theta_2D_{392}+u_2D_{1928}(D_{2534}+1)-u_1u_2(D_{168}+1)(D_{2534}+1)+u_2u_4D_{475}D_{1928},$$
$$D_E=(\theta_3-1)\theta_3-u_3D_{365}D_{392}+u_2u_3D_{365}D_{1928}-u_3u_5D_{392}D_{5867}$$
$$-u_1u_2u_3(D_{168}+1)D_{365}-u_3u_5u_7D_{392}(D_{798}+1)+u_2u_3u_4u_5D_{1928}D_{5867},$$
$$D_F=(\theta_3-\theta_4)(D_{2534}+1)+u_3D_{365}D_{392}-u_4D_{142}D_{475}.$$

Here Euler operators $\theta_j=u_j\partial_{u_j}$ are packaged into

$$D_{ijk}=\theta_i+\theta_j-\theta_k, \qquad\qquad D_{ijkl}=\theta_i+\theta_j-\theta_k-\theta_l.$$

Big step from box (2 cross ratios) to double box (9 cross ratios)!

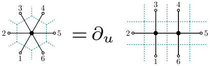
# Massless Double Box and Hexagon

Solution of Yangian constraints yields basis of 4834 series (double box) or 2530 series (hexagon) compared to 12 for the box integral:

$$H_k = \sum_{n_j \in x_j + \mathbb{Z}} h_{n_1,\ldots,n_9} w_1^{n_1} \ldots w_9^{n_9}.$$

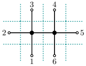
Mellin–Barnes approach yields linear combinations [Ananthanarayan, Banik, Friot, Ghosh 2020 + details announced]

Hexagon (26 Series):




$$\phi_6 = \frac{Q_6}{\Gamma(D/2-f)\Gamma(f)}(H_1 + H_4 + H_9 + H_{15} + H_{22} + H_{35} + H_{58} + H_{94} + H_{103} + H_{123} + H_{133} + H_{199} + H_{210} + H_{270} + H_{331} + H_{409} + H_{637} + H_{653} + H_{838} + H_{925} + H_{960} + H_{1375} + H_{1664} + H_{1675} + H_{2062} + H_{2442})$$

Double Box (44 Series):



$$\phi_{3,3} = \frac{Q_{3,3}\, u_8^{D/2-\ell}}{\Gamma(D/2-f)\Gamma(f)}(D_1 + D_4 + D_{10} + D_{18} + D_{27} + D_{45} + D_{76} + D_{140} + D_{158} + D_{190} + D_{208} + D_{318} + D_{340} + D_{440} + D_{542} + D_{674} + D_{1063} + D_{1091} + D_{1435} + D_{1581} + D_{1646} + D_{2382} + D_{3047} + D_{3068} + D_{3786} + D_{4580} + D_5 + D_{19} + D_{28} + D_{46} + D_{142} + D_{210} + D_{826} + D_{926} + D_{942} + D_{988} + D_{1094} + D_{1112} + D_{1330} + D_{1449} + D_{1647} + D_{2436} + D_{3069} + D_{3806})$$

What are the best variables? Refine algorithm on simpler examples?

# Yangian Symmetry for the Masses

# Introduce Masses

In the massless case, $\mathcal{N}=4$ SYM theory was the starting point:

$\mathcal{N}=4$ SYM $\rightarrow$ $\gamma$-deformation $\rightarrow$ fishnet theory $\rightarrow$ Feynman graphs

Introduce masses into $\mathcal{N}=4$ SYM by giving VEV to one of the scalars:

$$\hat{\Phi} = \langle\Phi\rangle + \Phi$$

Leads to massive propagators with difference mass ($x_{jk}^\mu = x_j^\mu - x_k^\mu$):

$$1/x_{jk}^2 \quad \rightarrow \quad 1/\hat{x}_{jk}^2 = 1/(x_{jk}^2 + (m_j - m_k)^2)$$

Well known massive dual conformal symmetry generated by: [Alday, Henn 2010; Plefka, Schuster]

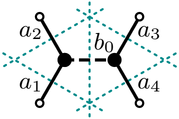
$$\mathrm{P}_j^{\hat{\mu}} = -i\partial_{x_j}^{\hat{\mu}}, \qquad \mathrm{L}_j^{\hat{\mu}\hat{\nu}} = ix_j^{\hat{\mu}}\partial_{x_j}^{\hat{\nu}} - ix_j^{\hat{\nu}}\partial_{x_j}^{\hat{\mu}},$$
$$\mathrm{D}_j = -i\big(x_{j\mu}\partial_{x_j}^{\mu} + m_j\partial_{m_j} + \Delta_j\big),$$
$$\mathrm{K}_j^{\hat{\mu}} = -2ix_j^{\hat{\mu}}\big(x_{j\nu}\partial_{x_j}^{\nu} + m_j\partial_{m_j} + \Delta_j\big) + i(x_j^2 + m_j^2)\partial_{x_j}^{\hat{\mu}}.$$

Mass interpreted as *fifth dimension* (radial direction in AdS): $x^{\hat{\mu}=5} = m_j$.

**...but no massive Yangian symmetry known.**

# Massive Feynman Integrals

Consider massive Feynman integrals directly, e.g.




$$= \int \frac{\mathrm{d}^D x_0 \mathrm{d}^D x_{\bar{0}}}{\hat{x}_{01}^{2a_1} \hat{x}_{02}^{2a_2} x_{0\bar{0}}^{2b_0} \hat{x}_{\bar{0}3}^{2a_3} \hat{x}_{\bar{0}4}^{2a_4}},$$

Use massive dual conformal (level-zero) generators $\mathrm{J}^a$ to build Yangian level-one generators:

$$\widehat{\mathrm{J}}^a = \tfrac{1}{2} f^a{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}_j^c \mathrm{J}_k^b + \sum_{j=1}^{n} s_j \mathrm{J}_j^a,$$

with e.g. the level-one momentum generator

$$\widehat{\mathrm{P}}^{\hat{\mu}} = \tfrac{i}{2} \sum_{j,k=1}^{n} \mathsf{sign}(k-j) \big( \mathrm{P}_j^{\hat{\mu}} \mathrm{D}_k + \mathrm{P}_{j\nu} \mathrm{L}_k^{\nu\hat{\mu}} \big) + \sum_{j=1}^{n} s_j \mathrm{P}_j^{\hat{\mu}},$$

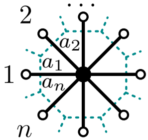
Note: We split off internal sum over $D+1$-component:

$$\widehat{\mathrm{P}}^{\hat{\mu}}_{\mathsf{extra}} = \tfrac{i}{2} \sum_{j<k} \mathsf{sign}(k-j) \mathrm{P}_{j,D+1} \mathrm{L}_k^{D+1,\hat{\mu}}.$$

# One Loop

Generic one-loop integral:

$$I_n = \int \mathrm{d}^D x_0 \prod_{j=1}^{n} (x_{0j}^2 + m_j^2)^{-a_j} =$$



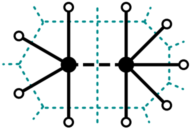

We find the general statement

$$\widehat{\mathrm{J}}^a I_n = 0.$$

- works for massless or massive propagators,
- at one loop: even integrand invariant under two-point density $\widehat{\mathrm{J}}^a_{jk}$,
- extra-dimensional contribution $\widehat{\mathrm{J}}^a_{\text{extra}}$ is an extra symmetry.

# Two Loops

We can show that two-loop integrals with massless internal propagators

$I_{lr} =$ 

are Yangian invariant, i.e. $\widehat{\mathrm{J}}^a I_{lr} = 0$, using:

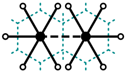
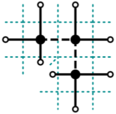
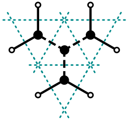
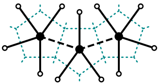
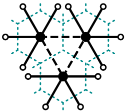
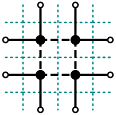
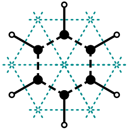
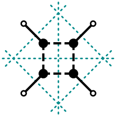
- level-zero invariance,
- the Yangian invariance of one-loop integrals.

**Note:** In the massless limit we have Yangian symmetry for all planar one- and two-loop integrals with non-coincident external points.

# Higher Loops

**Conjecture:**

All planar Feynman graphs, which are cut along a closed contour from one of the three regular tilings of the plane, have massive Yangian symmetry if all internal propagators are massless. External propagators can be massive or massless.

| Yangian Symmetry | | | No Yangian |
|---|---|---|---|
| Triangle | Square | Hexagon | Irregular |
|  |  |  |  |
|  |  |  |  |

Supported by numerical evidence.

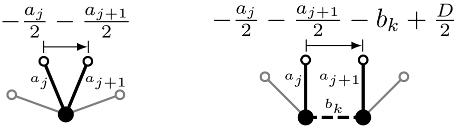
# Evaluation Parameters

Parameters $s_j$ enter definition of level-one generators:

$$\widehat{\mathrm{J}}^a = \tfrac{1}{2} f^a{}_{bc} \sum_{j<k=1}^{n} \mathrm{J}_j^c \mathrm{J}_k^b + \sum_{j=1}^{n} s_j \mathrm{J}_j^a,$$

**Prescription:**

- choose (arbitrary) leg 1 and set $s_1 = 0$,
- get $s_{j+1}$ from $s_j$ by adding e.g.



# Summary of Massive Symmetries

| Loops | Graphs | Dual Conformal | Not Dual Conformal |
|---|---|---|---|
| One | $n$-gons | Full Yangian & $\widehat{\mathrm{J}}^a_{\text{extra}}$ | All $\widehat{\mathrm{J}}^a$ & $\widehat{\mathrm{J}}^a_{\text{extra}}$ |
| Two | $l$-$r$-gons | Full Yangian | $\widehat{\mathrm{P}}^\mu$ |
| All | Tilings | Full Yangian | $\widehat{\mathrm{P}}^\mu$ |

Dual conformal means $D = \sum_{k \in \text{vertex}} a_k$

**Note:** We can have level-one symmetry even without full dual conformal level-zero symmetry.

# Momentum Space Symmetry

In the massless case, the Yangian algebra can be understood as the closure of dual and ordinary conformal symmetry [Drummond, Henn Plefka, 2009]

Translate level-one generators in $x$-space into dual momentum space:

$$p_j^\mu = x_j^\mu - x_{j+1}^\mu$$

For $x$-space level-one momentum generator one finds

$$\widehat{\mathrm{P}}^\mu = -\tfrac{i}{2}\bar{\mathrm{K}}^\mu\,,$$

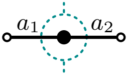
where $\bar{\mathrm{K}}^\mu$ forms part of novel massive generalization of momentum space conformal symmetry:

$$\begin{aligned}
\bar{\mathrm{P}}_j^\mu &= p_j^\mu\,, \qquad \bar{\mathrm{L}}_j^{\mu\nu} = p_j^\mu \partial_{p_j}^\nu - p_j^\nu \partial_{p_j}^\mu\,,\\
\bar{\mathrm{D}}_j &= p_{j\nu}\partial_{p_j}^\nu + \tfrac{m_j\partial_{m_j} + m_{j+1}\partial_{m_{j+1}}}{2} + \bar{\Delta}_j\,,\\
\bar{\mathrm{K}}_j^\mu &= p_j^\mu \partial_{p_j}^2 - 2\Big[p_{j\nu}\partial_{p_j}^\nu + \tfrac{m_j\partial_{m_j} + m_{j+1}\partial_{m_{j+1}}}{2} + \bar{\Delta}_j\Big]\partial_{p_j}^\mu\,.
\end{aligned}$$

# Bootstrap Example I: Bubble

Consider dual conformal integral with $v = \frac{m_1^2+m_2^2+x_{12}^2}{2m_1m_2}$ and $a_1 + a_2 = D$:

$$I_2^{m_1 m_2} =$$




$$= \frac{(1-v^2)^{\beta/2}}{m_1^{a_1} m_2^{a_2}}\phi(v),$$

Yangian PDE reads ($2\alpha = a_1 - a_2 - 1$ and $2\beta = -a_1 - a_2 + 1$)

$$\left[\alpha(\alpha+1) + \tfrac{\beta^2}{v^2-1}\right]\phi - 2v\phi' + (1-v^2)\phi'' = 0.$$

Solved by associated Legendre functions of the first and second kind:

$$\phi(v) = c_1 P_\alpha^\beta(v) + c_2\, Q_\alpha^\beta(v).$$

Numerical data points yield coefficients

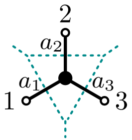
$$\phi(v) = 2^\beta \pi^{1-\beta} P_\alpha^\beta(v).$$

Integral fixed from symmetries.

# Bootstrap Example II: Triangle

Consider dual conformal triangle integral:

$$I_3^{m_1 m_2 m_3} =$$




$$= \frac{\phi_3(u,v,w)}{m_1^{a_1} m_2^{a_2} m_3^{a_3}},$$

with $u = u_{12}, \quad v = u_{13}, \quad w = u_{23}, \qquad u_{jk} = -\hat{x}_{jk}^2/4m_j m_k.$

Straightforward to solve Yangian PDEs with series ansatz:

$$H_C(u,v,w) = \sum_{k,l,n=0}^{\infty} \frac{(a_1)_{k+l}(a_2)_{k+n}(a_3)_{l+n}}{(\gamma)_{k+l+n}} \frac{u^k}{k!} \frac{v^l}{l!} \frac{w^n}{n!},$$

[Srivastava]

Overall coefficient found from numerics:

$$I_3^{m_1 m_2 m_3} = \frac{\pi^{D/2} \Gamma_{D/2}}{(D-1)! m_1^{a_1} m_2^{a_2} m_3^{a_3}} H_C(u,v,w).$$

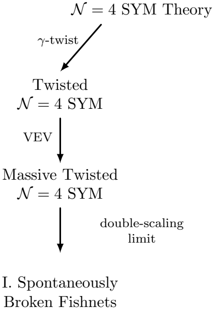
Integral fixed from symmetries.

# Massive Fishnet Theory

# Where Does Massive Integrability Come From?

? → massive Feynman graphs

Can we define a massive version of the fishnet theory?




First idea: Spontaneous Symmetry Breaking in the massless Fishnet Theory

- product-mass propagators $p^2 + m_j m_k$
- not the Feynman graphs above

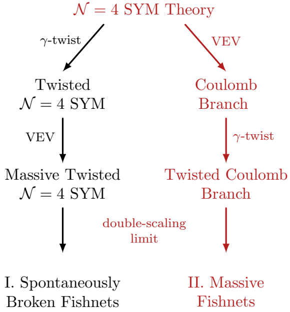
but interesting for different reasons:

- 'natural' symmetry breaking [Karananas, Kazakov Shaposhnikov 2019]
- soft theorems for amplitudes [FL, Miczajka 2020]

# Where Does Massive Integrability Come From?

? $\rightarrow$ massive Feynman graphs

Can we define a massive version of the fishnet theory?




Second idea: Double-scaling limit of Coulomb branch $\mathcal{N}=4$ SYM theory

- similar route as in massless case
- yields difference mass propagators $p^2+(m_j-m_k)^2$

# Double-Scaling Limit of Coulomb-Branch $\mathcal{N} = 4$ SYM

$\mathcal{L}_{\text{Coul}} \xrightarrow{e^{i\gamma}} \mathcal{L}^{\gamma}_{\text{Coul}} \to$ double-scaling limit $\to$ massive Feynman graphs

Critical step: $\gamma$-deformation

- Phase operator only well-defined on R-symmetry singlets:

$$\mathcal{P}_\gamma = \exp\bigl(\tfrac{i}{2}\gamma_A \epsilon_{ABC}\, q^B \wedge q^C\bigr)$$

- Ad hoc solution: average first over ways to break up trace:

$$Q : \mathrm{tr}(\Phi_1\Phi_2 \ldots \Phi_n) \mapsto \tfrac{1}{n}\bigl(\Phi_1\Phi_2 \ldots \Phi_n + \Phi_2\Phi_3 \ldots \Phi_1 + \ldots\bigr)$$

- and define deformation as

$$\mathcal{L}^{\gamma}_{\text{Coul}} = Q^{-1}\mathcal{P}_\gamma Q \mathcal{L}_{\text{Coul}}.$$

Then proceed with double-scaling limit as in massless case ...

# Massive Fishnet Theory

Most restrictive limit results in massive bi-scalar theory: [FL, Miczajka 2020]

$$\mathcal{L}_{\mathsf{MF}} = N_{\mathsf{c}}\,\mathrm{tr}\big(-\partial_\mu \bar{X}\partial^\mu X - \partial_\mu \bar{Z}\partial^\mu Z + \xi^2 \bar{X}\bar{Z}XZ\big)$$
$$- N_{\mathsf{c}}(m_a - m_b)^2 X^a{}_b \bar{X}^b_a - N_{\mathsf{c}}(m_a - m_b)^2 Z^a{}_b \bar{Z}^b_a.$$

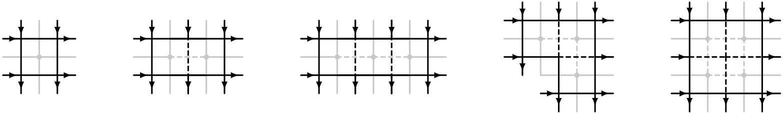
Planar Limit:

- Choose diagonal VEV with finitely many non-vanishing entries:

$$V = \mathsf{diag}(v_1, \ldots, v_n, 0, 0 \ldots, 0), \qquad v_a \sim m_a.$$

Integrability:

- Planar off-shell amplitudes in one-to-one correspondence with Yangian-invariant massive Feynman integrals in 4D, e.g.



**Note:** Further limit-theories can be obtained in this way.

# Tri-Scalar Theory

Consider alternative limit: [FL, Miczajka 2020]

$$\xi_1^2 = g e^{-i\gamma_1}, \qquad \xi_2^2 = g e^{-i\gamma_2}, \qquad \xi_3^2 = g^2 e^{-i\gamma_3},$$

and then take

$$g \to 0, \qquad \gamma_j \to i\infty, \qquad \xi_j = \text{fix}.$$

which yields tri-scalar theory:

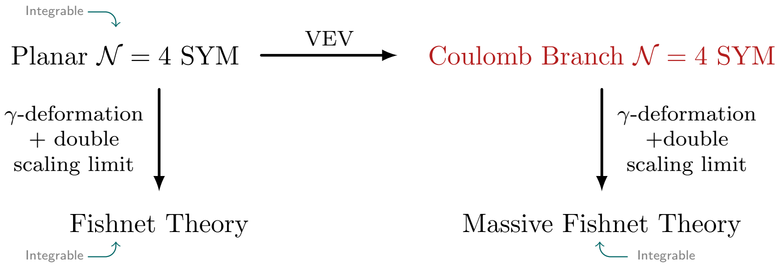
$$\mathcal{L}_{\mathsf{F}}^{\mathsf{Tri}} = N_{\mathsf{c}}\,\mathrm{tr}\big(-\partial_\mu \bar{X}\partial^\mu X - \partial_\mu \bar{Z}\partial^\mu Z - \tfrac{1}{2}\partial_\mu Y \partial^\mu Y + \xi_3^2 \bar{X}\bar{Z}XZ\big)$$

with mass contribution ($\Lambda = \mathrm{diag}(m_1, \dots, m_{N_{\mathsf{c}}})$)

$$\mathcal{L}_{\mathsf{M}}^{\mathsf{Tri}} = N_{\mathsf{c}}\,\mathrm{tr}\Big[[\Lambda, X][\Lambda, \bar{X}] + [\Lambda, Z][\Lambda, \bar{Z}] + \tfrac{1}{2}[\Lambda, Y]^2 + \frac{\xi_1^2}{\sqrt{2}}\big(\Lambda Z Y \bar{Z} - \tfrac{1}{4}\Lambda Y \bar{Z} Z - \tfrac{1}{4}\Lambda \bar{Z} Z Y\big) + \frac{\xi_2^2}{\sqrt{2}}\big(\Lambda \bar{X} Y X - \tfrac{1}{4}\Lambda Y X \bar{X} - \tfrac{1}{4}\Lambda X \bar{X} Y\big)\Big].$$

Would be interesting to analyze this theory with regard to integrability...

# Integrability on the Coulomb Branch?



# Summary

- Massless integrals inherit Yangian symmetry from $\mathcal{N} = 4$ SYM.
- Surprising integrability of massive Feynman integrals.
- Massive generalization of momentum space conformal symmetry.
- Allows to bootstrap Feynman integrals from scratch.
- Leads to integrable massive fishnet theory.

# Outlook

- Proof conjectured symmetry at higher loops?
- How efficient is Yangian bootstrap for higher loop integrals?
- Double-Scaling limit yields further massive fishnet theories. Are these integrable?
- Massive fishchain?
- Is $\mathcal{N}=4$ SYM theory on the Coulomb branch integrable?

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Thanks!