Integrability from braided tensor categories

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At long last **2008.02292**

Closely related work on lattice topological defects **2008.08598** (Ising in arXiv:**1601.07185)**

with David Aasen (Station Q) and Roger Mong (Pittsburgh)

Many interesting two-dimensional statistical mechanical models are conveniently described in terms of knot and link invariants.

This observation was made in the ’80s, but mostly neglected since.

The excitement about topological quantum computation ignited condensed-matter physicists’ interest in knot invariants, and the time is ripe for (re)applying them to statistical mechanics.

The key mathematics I exploit is that a tensor category gives a consistent set of rules that allow graphs to be manipulated without changing the topological invariants.

The moral of the story is: draw pictures!

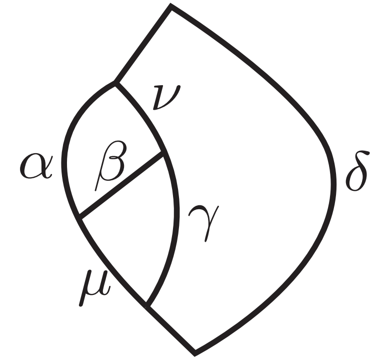
Outline

1. Fusion categories
2. Defining lattice models using fusion categories
   - Geometric (e.g. counting loops)
   - Spins/heights (e.g. Ising/Potts/hard squares/RSOS)
   - Quantum Hamiltonians (e.g. spin/anyon chains)
3. Integrable lattice models via Yang-Baxter
4. Braided tensor categories
5. Baxterization via fractional-spin conserved currents

Will give a simple formula for the Boltzmann weights in terms of category data that guarantees the existence of conserved currents. In all known examples, these weights also satisfy the Yang-Baxter equation.

1. Fusion categories

Associates a topological invariant to a fusion diagram, a labeled planar graph built from trivalent vertices.

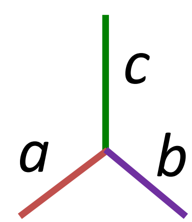


The edges are labeled by the (finite number of) objects a, b, c, …

Non-negative integer

The objects satisfy a fusion algebra $a \otimes b = \bigoplus_c N_{ab}^c \, c$

For each $N_{ab}^c \neq 0$, a trivalent vertex



For example, the tensor products of representations of (the quantum deformation of) *SO*(*n*) form a fusion algebra. Identity rep is equivalent to no line.

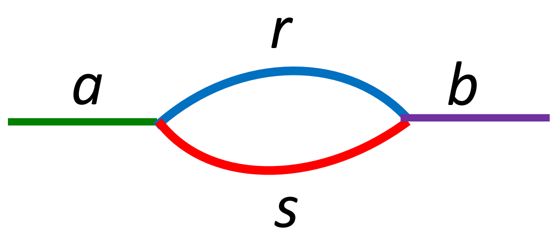
Evaluating the fusion diagram

is done using data from a category $\mathcal{C}$:

quantum dimensions d_a

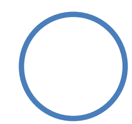
F symbols: $F_{tt'}\begin{bmatrix} r & s \\ a & b \end{bmatrix}$

bubble removal:



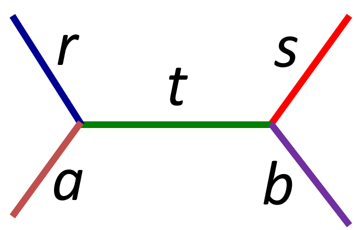
$= \delta_{ab}\sqrt{\frac{d_r d_s}{d_a}}$

Setting *a*=*b*=0 allows a closed loop to be removed completely:

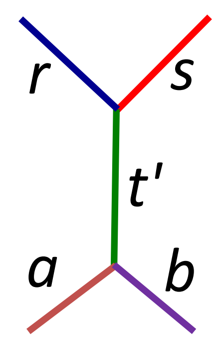


$= d_r$

F move:

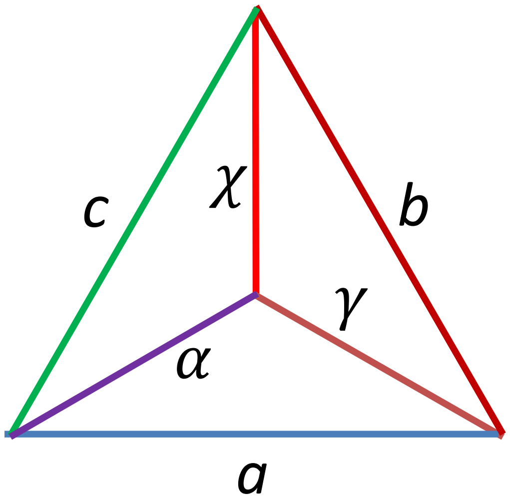


$= \sum_{t' \in \mathcal{C}} F_{tt'}\begin{bmatrix} r & s \\ a & b \end{bmatrix}$



F moves preserve the evaluations. Doing them and bubble removal repeatedly express a fusion diagram as a sum over closed loops, and hence a number.

Lots of identities and consistency relations



$$= \sqrt{d_\alpha d_\gamma d_b d_c}\, F_{a\chi}\begin{bmatrix}\alpha & \gamma \\ c & b\end{bmatrix} = \sqrt{d_a d_\gamma d_\chi d_c}\, F_{b\alpha}\begin{bmatrix}\chi & \gamma \\ c & a\end{bmatrix}$$

A famous consistency relation is the pentagon equation, which is of the form

$$F\,F = \sum F\,F\,F$$

I view the category and its data as input to the manipulations.

2. Lattice models from fusion categories

The story long predates fusion categories:

Proc. Roy. Soc. Lond. A. **322**, 251–280 (1971)
Printed in Great Britain

Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem

By H. N. V. Temperley
Department of Applied Mathematics, University College, Swansea, Wales, U.K.

and E. H. Lieb†
Department of Mathematics, Massachussetts Institute of Technology, Cambridge, Mass., U.S.A.

Aka Tutte polynomial
Aka Potts model partition function

A transfer-matrix approach is introduced to calculate the 'Whitney polynomial' of a planar lattice, which is a generalization of the 'percolation' and 'colouring' problems. This new approach turns out to be equivalent to calculating eigenvalues and traces of Heisenberg type operators on an auxiliary lattice which are very closely related to problems of 'ice' or 'hydrogen-bond' type that have been solved analytically by Lieb (1967*a* to *d*). Solutions for

aka six-vertex model/XXZ chain

Temperley and Lieb showed how the transfer matrices of seemingly different models could be written in terms of operators obeying the same algebra.

With appropriate choices of boundary conditions, their partition functions can be related exactly.

To relate to loop and to local models, another innovation was required:

ON THE RANDOM-CLUSTER MODEL

I. INTRODUCTION AND RELATION TO OTHER MODELS

C. M. FORTUIN and P. W. KASTELEYN

Instituut Lorentz, Rijksuniversiteit te Leiden, Nederland

Received 5 July 1971

The loops surround the clusters

Synopsis

The random-cluster model is defined as a model for phase transitions and other phenomena in lattice systems, or more generally in systems with a graph structure. The model is characterized by a (probability) measure on a graph and a real parameter κ. By specifying the value of κ to 1, 2, 3, 4, ... it is shown that the model covers the percolation model, the Ising model, the Ashkin–Teller–Potts model with 3, 4, ... states per atom, respectively, and thereby, contains information on graph-colouring problems; in the limit $\kappa \downarrow 0$ it describes linear resistance networks. It is shown that the function which for the random-cluster model plays the role of a partition function, is a generalization of the dichromatic polynomial earlier introduced by Tutte, and related polynomials.

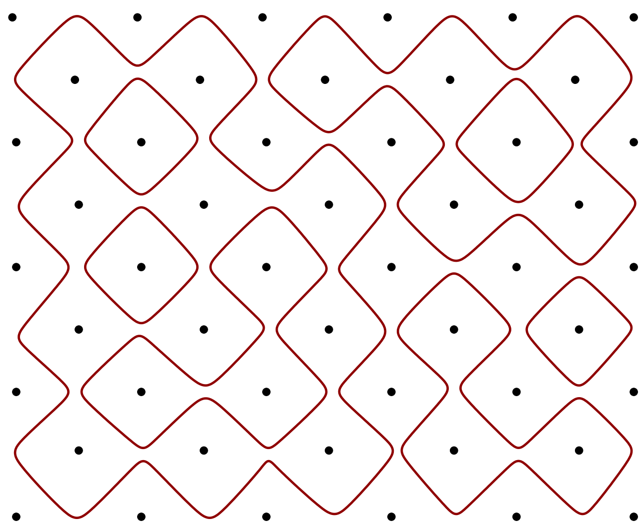
These results can be systematized and generalized by defining Boltzmann weights using a fusion category.

Three classes of models, all closely related:

- Geometric (non-local weights, e.g. counting loops)
- Spins/heights (local weights, e.g. Ising/Potts/hard squares)
- Quantum Hamiltonians (e.g. spin/anyon chains)

The completely packed loop/ random-cluster/Q-state Potts models

Every edge of the square lattice is covered by non-crossing loops; the only degrees of freedom are how they avoid at each vertex.



$$Z = \sum_{\substack{\text{loop} \\ \text{configurations}}} d^{\#\text{loops}} v^{\# >\!<} h^{\# \genfrac{}{}{0pt}{}{\vee}{\wedge}}$$

d is the weight per loop,

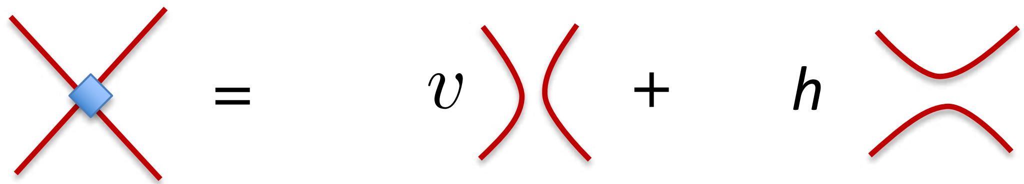
v the weight per vertical avoidance,

h the weight per horizontal avoidance

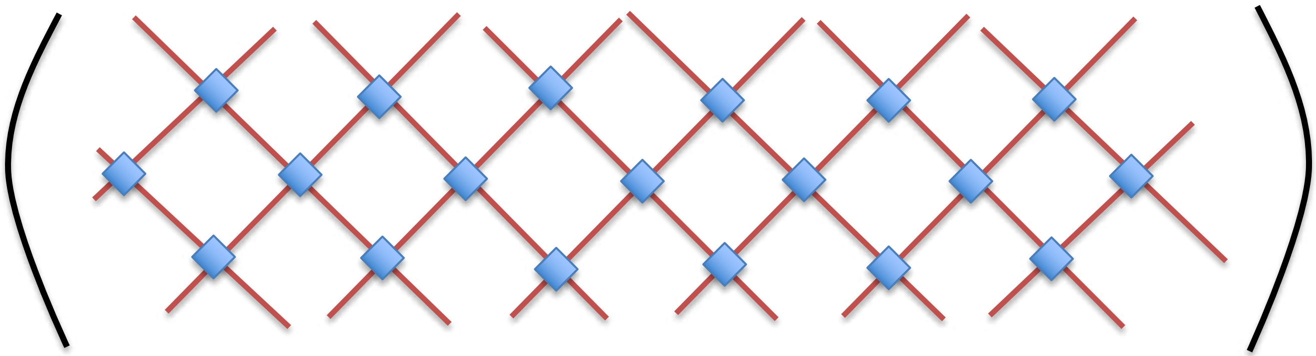
Takes form $Z = \sum_{\text{loops}}$ (topological weight) x (local weights)

The Boltzmann weights, pictorially

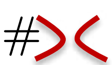
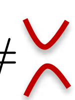
Picture loop model Boltzmann weights on the square lattice as



$Z =$ eval



where eval means to expand out each vertex to get loops with the appropriate local weights, then sum over all loop configurations with weight *d* per loop.

$Z = \sum_{\substack{\text{loop} \\ \text{configurations}}} d^{\#\text{loops}} v^{\#}$  $h^{\#}$ 

Loops and fusion categories

This loop model can be written in terms of the fusion category $SU(2)_k$

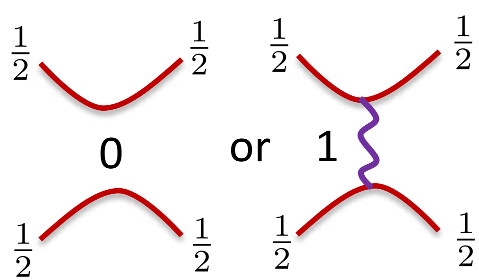
The k+1 objects are labelled $0,\ \frac{1}{2},\ 1,\ \ldots\ ,\ \frac{k}{2}$

They obey a truncated version of the fusion rules of the representations of $SL(2)$

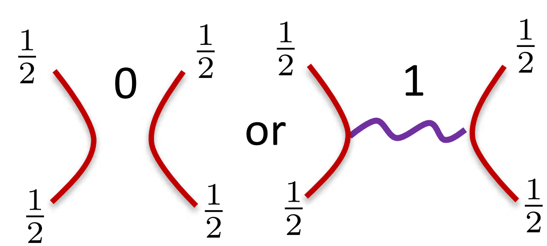
e.g. $$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

The lines making up the loops are labeled by the object $\frac{1}{2}$

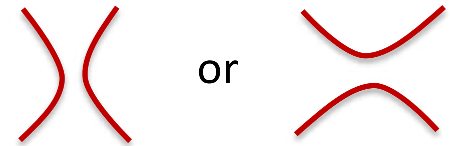
When two lines meet in the fusion category model, they fuse:



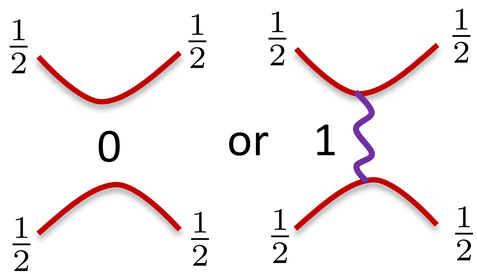
another basis:



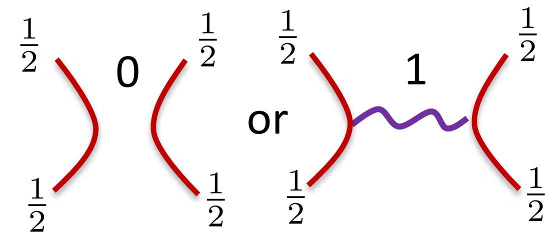
When two lines meet in the loop model, they avoid:



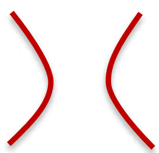
Basis 1:



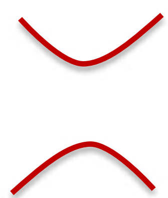
Basis 2:



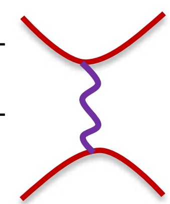
F moves give two linear relations among the four diagrams:

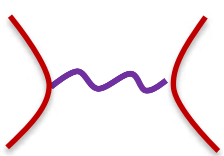


$= \frac{1}{d}$

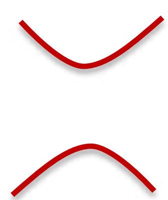


$+ \frac{\sqrt{d_1}}{d}$

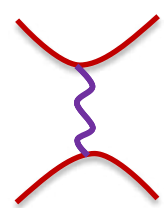




$= \frac{\sqrt{d_1}}{d}$



$- \frac{1}{d}$



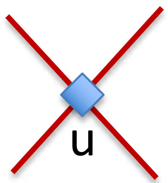
$$d = d_{\frac{1}{2}} = 2\cos\frac{\pi}{k+2}$$

$$d_1 = d^2 - 1 = \frac{\sin\frac{3\pi}{k+2}}{\sin\frac{\pi}{k+2}}$$

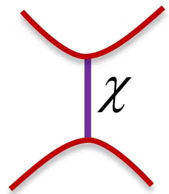
Can rewrite degrees of freedom in any fashion desired.

Advantages of using categories

- Can define geometric models for any category, any label on the lines:



$$= \sum_{\chi} A_{\chi}(u)$$



$$Z = \sum_{\mathcal{F}} \text{eval}\left[\mathcal{F}\right] \times \text{local weights}$$

Local weights depend on ``spectral parameter'' u

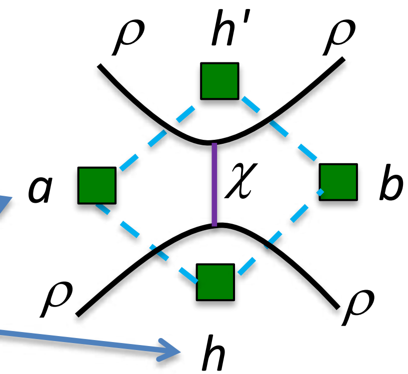
- Allows local models to be rewritten in terms of topological data

- Derive new properties, even for ancient models like the Ising and Potts models

Topological structure of local models

``Height'' degrees of freedom live on the graph dual to the fusion diagram

Heights are objects in the category

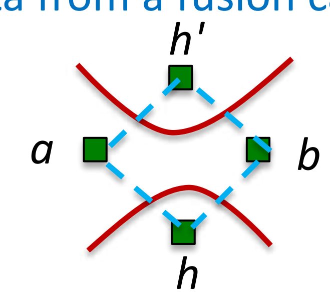


$$= \frac{d_\rho}{\sqrt{d_\chi}} F_{h\chi}\begin{bmatrix}\rho & \rho\\ a & b\end{bmatrix} F_{\chi h'}\begin{bmatrix} a & \rho \\ b & \rho\end{bmatrix}$$

Through the miracle of shadow world, can define local Boltzmann weights for the heights so that the partition function is the same as the corresponding geometric model!

Temperley-Lieb, Baxter-Kelland-Wu, Pasquier, Jones, Reshetikhin, Turaev-Viro, Barrett-Westbury …

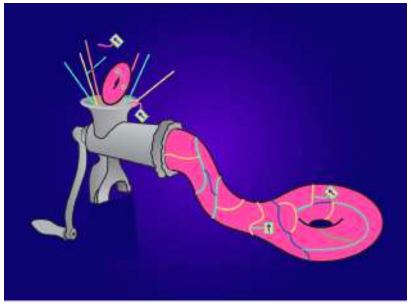
Using shadow world, critical Ising, Potts, RSOS, 8-vertex, … models all can be written in terms of topological data from a fusion category, e.g.



$$= \delta_{ab}\sqrt{\frac{d_h d_{h'}}{d_a d_b}}$$

Lattice topological defects

Fusion categories give a general and systematic way of finding lattice topological defects with many nice properties.

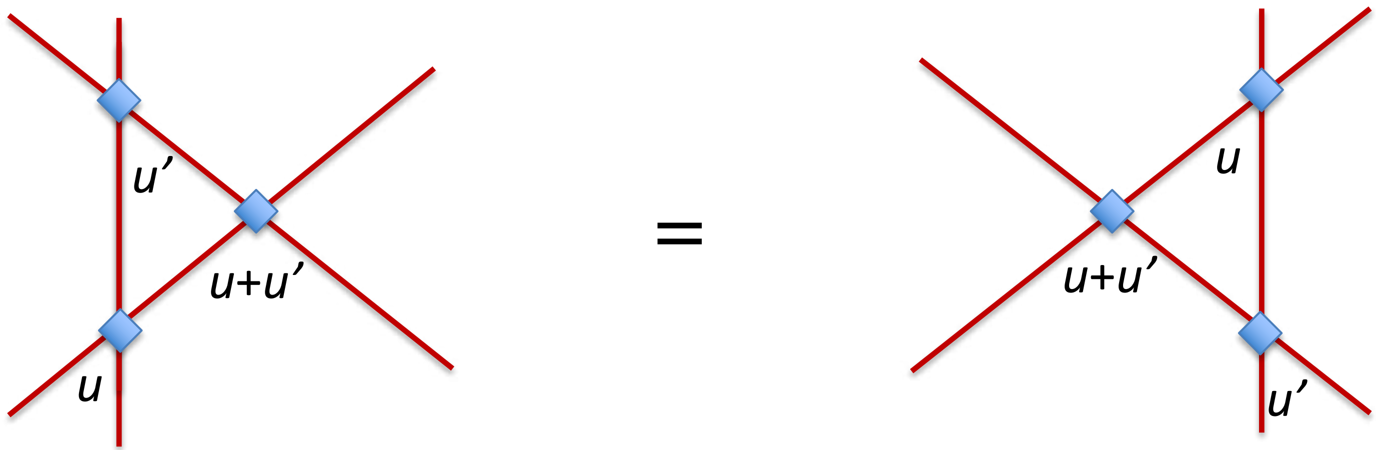


- Topologically invariant junctions of defect lines
- Many generalisations of Kramers-Wannier duality.
- Understand exact degeneracies in gapped systems
- Exact lattice derivation of ratios of boundary *g*-factors
- By doing modular transformations, get momentum quantization conditions that yield exact lattice results for conformal spins, giving strong constraints on any continuum limit.

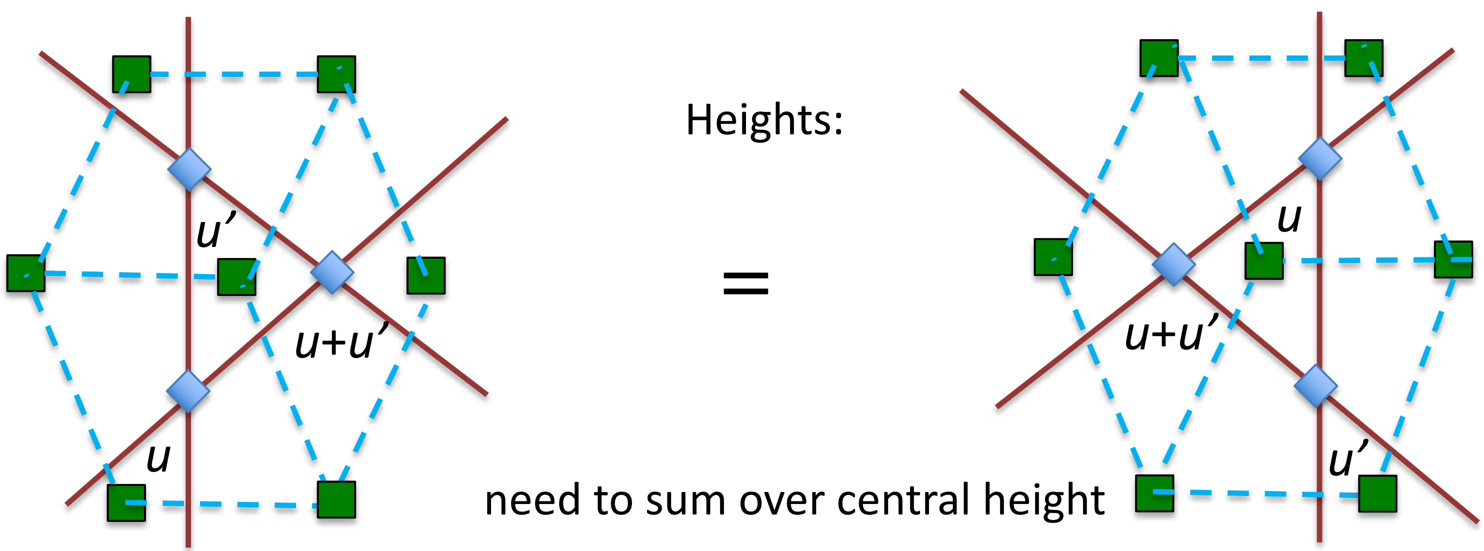
Aasen, Fendley and Mong

3.Integrability from the Yang-Baxter equation

Sums of products of three Boltzmann weights obey



Note *u* and *u'* have changed places: use to construct commuting transfer matrices and the resulting local conserved currents needed for integrability.



The YBE for completely packed loops

The YBE gives functional equations for the Boltzmann weights.

For the loop model above, get for the local weight ratio $w(u) = \frac{h(u)}{v(u)}$

$$w(u)w(u+u')w(u') + d\, w(u)w(u') + w(u) + w(u') - w(u+u') = 0$$

Parametrize the weight per loop by $d = q + q^{-1}$. Then the solution is

$$w(u) = \frac{q^{-1}e^{iu} - qe^{-iu}}{e^{iu} - e^{-iu}}$$

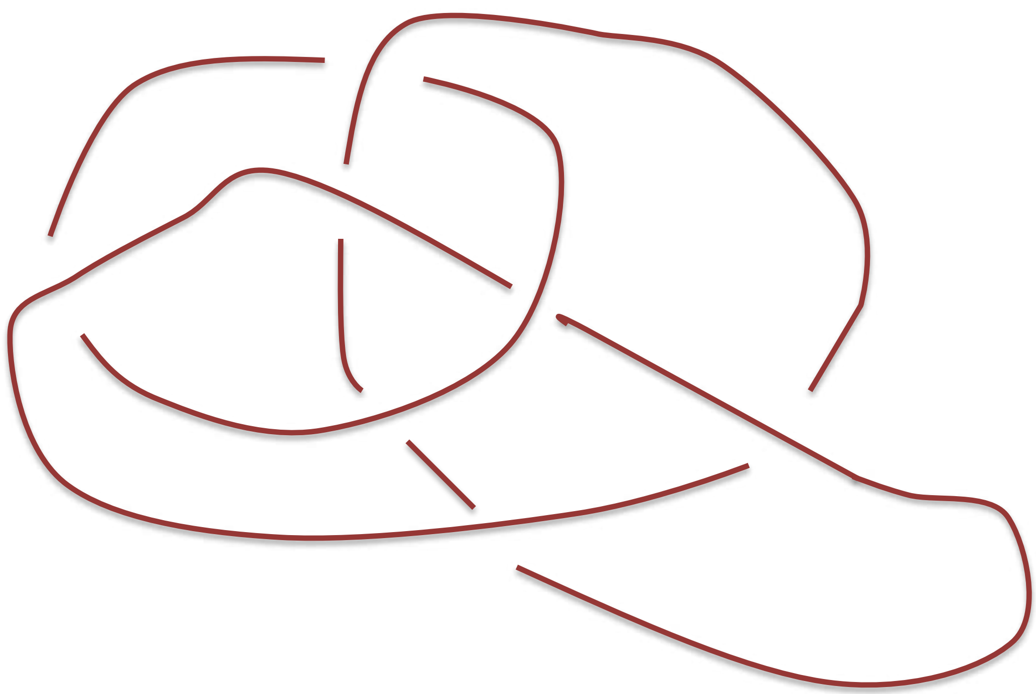
Using shadow world gives local Boltzmann weights for the Andrews-Baxter-Forrester height models at their integrable critical points.

How does something so simple arise from such a complicated equation?

4. Braiding

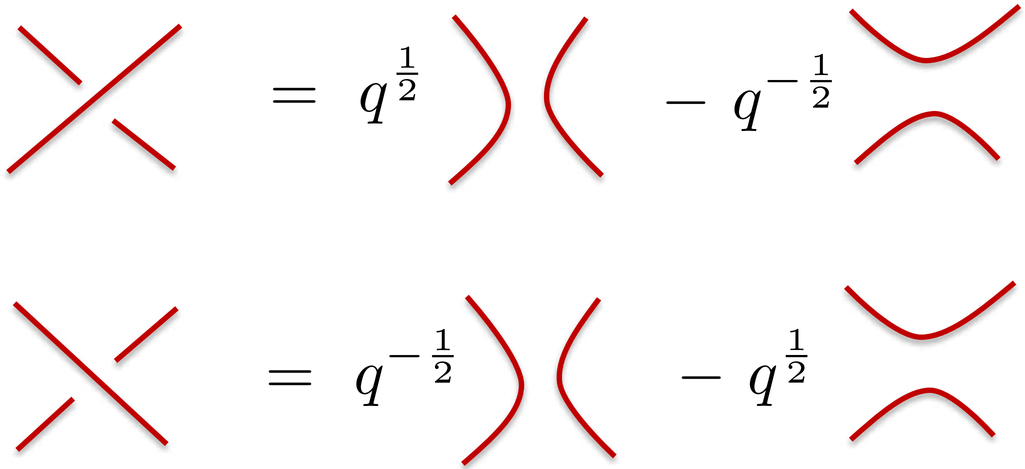
From a very lowbrow perspective, the key innovation of Jones was to show that the Temperley-Lieb algebra (the $SU(2)_k$ fusion category) can be extended to give representations of the braid group.

A knot or link invariant such as the Jones polynomial depends only on the topology of the knot. To compute, project the knot/link onto the plane:

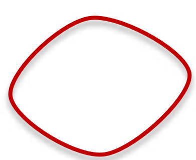


The skein relation resolves each over/undercrossing to turn each knot/link into a sum over planar fusion diagrams.

For the Jones polynomial, loops! The Kauffmann bracket:



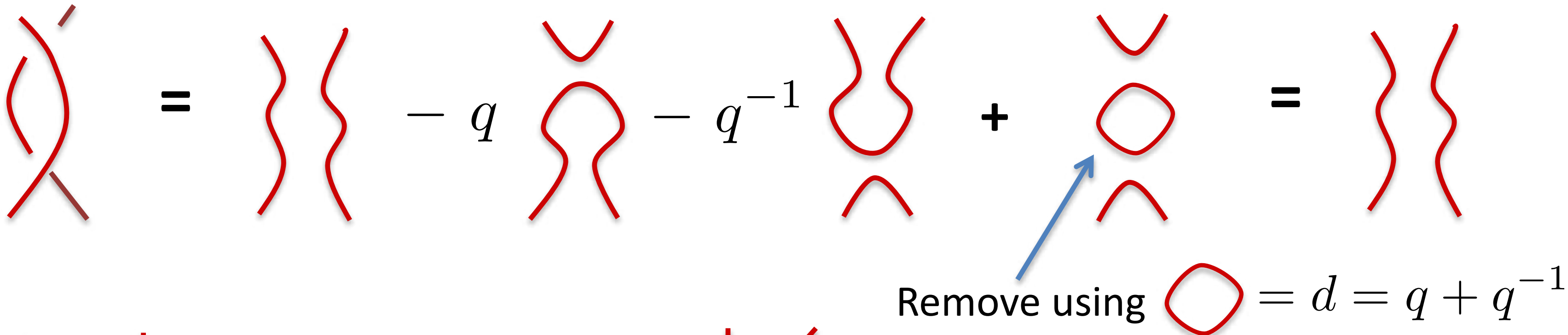
Resolving turns a link into a sum over graphs of closed loops. To get the Jones polynomial (in q), evaluate by replacing each loop with



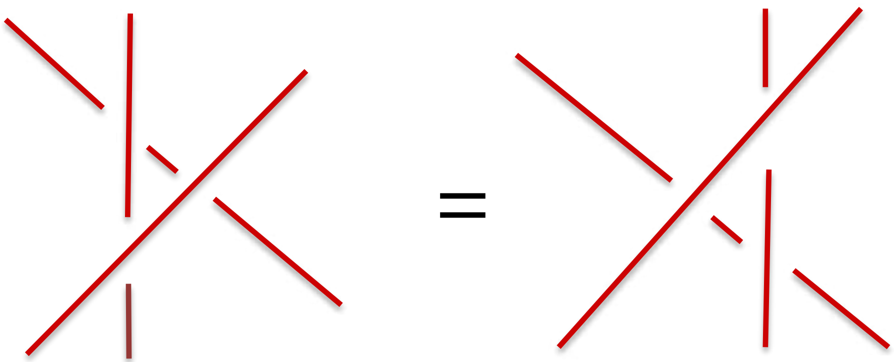
$$= d = q + q^{-1}$$

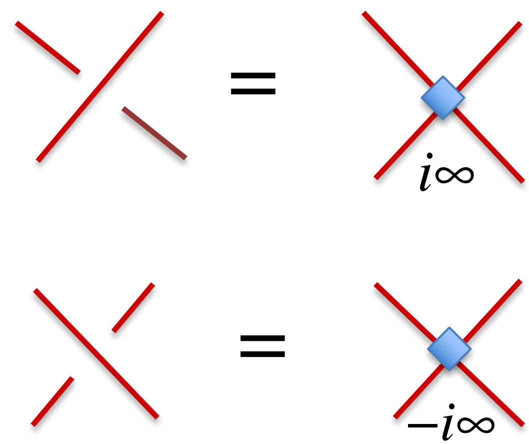
To yield a topological invariant, must satisfy the Reidemeister moves:

#2:

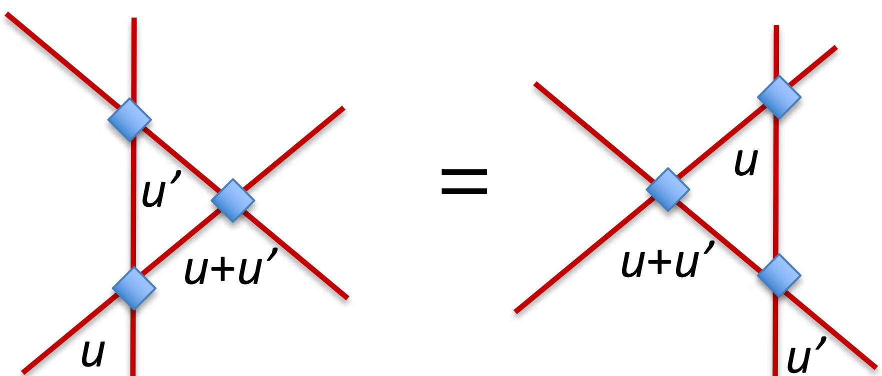


#3:



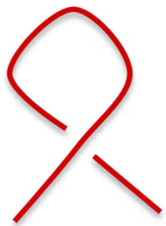


The resemblance to the YBE is obvious:

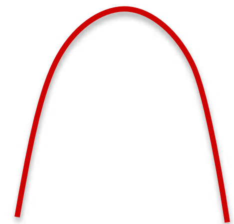


One subtlety turns out to be a feature, not a bug.

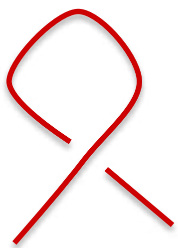
#1:



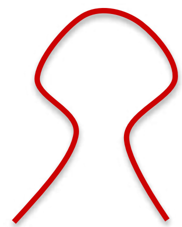
$=$



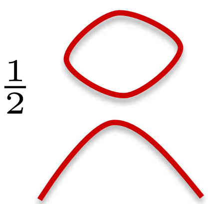
But instead:



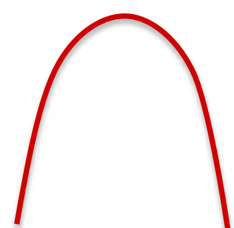
$= q^{\frac{1}{2}}$



$- q^{-\frac{1}{2}}$



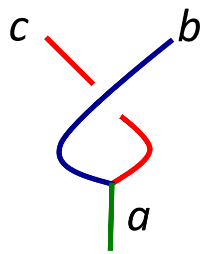
$= - q^{-\frac{3}{2}}$



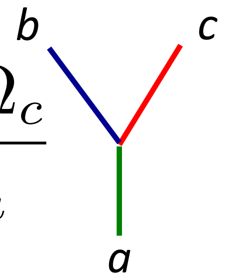
To undo, make each link a ribbon, and keep track of twists. Topological invariant after multiplying by $(-q)^{\frac{3}{2}w}$, where w = #(signed twists) = writhe.

Braided tensor categories

are fusion categories with more data, the spins s_a



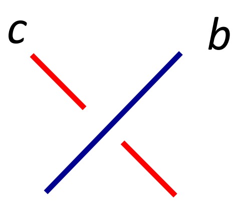
$$= \nu_{abc}\frac{\Omega_b\Omega_c}{\Omega_a}$$



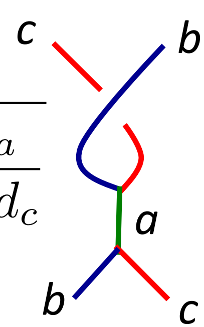
$$\nu_{abc} = \pm 1$$

$$\Omega_a = e^{i\pi s_a}$$

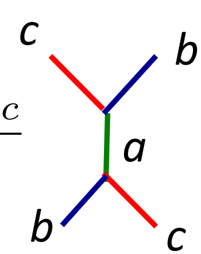
Don't need any more data, as can use this with *F* move to get braiding



$$= \sum_a \sqrt{\frac{d_a}{d_b d_c}}$$



$$= \sum_a \sqrt{\frac{d_a}{d_b d_c}}\ \frac{\Omega_b\Omega_c}{\Omega_a}$$



Baxterizing

The initial work by Jones prompted much work finding knot and link invariants from statistical-mechanical models in the mid to late ’80s.

Jones then suggested people try the converse: to Baxterize is to start with a knot invariant and then try to generalize to a solution of the YBE and hence a lattice integrable model.

Easier said than done, since the YBE is trilinear in the Boltzmann weights.

A somewhat successful approach came from exploiting the representation theory of quantum-group algebras.

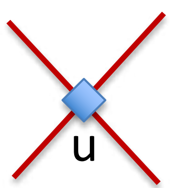
Jimbo; Zhang, Gould, Bracken, Delius

Their solution ends up involving very little of the quantum group.
Suggests there is a simpler and more general way...

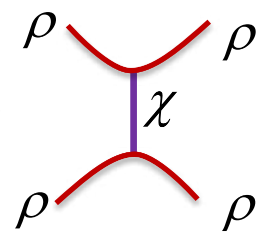
5. Baxterization from fractional-spin conserved currents

The category gives a natural and general way of defining Boltzmann weights that admit such currents.

The ensuing linear conditions on the weights can be solved easily.



$$= \sum_{\chi} A_\chi(u)$$



$$\frac{A_\chi(u)}{A_a(u)} = \frac{e^{iu}\Omega_\chi + \Omega_a}{\Omega_\chi + e^{iu}\Omega_a}$$

All known examples also satisfy Yang-Baxter!

QUANTUM GROUP SYMMETRIES IN TWO-DIMENSIONAL LATTICE QUANTUM FIELD THEORY

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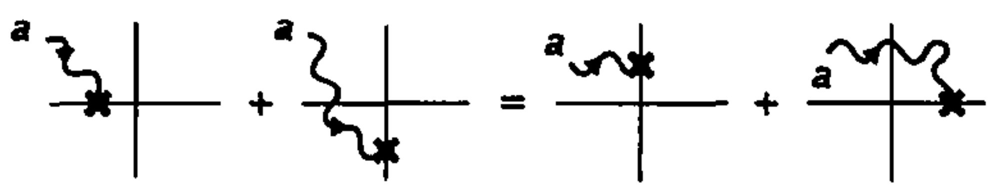
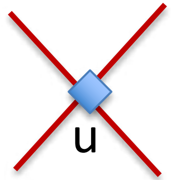
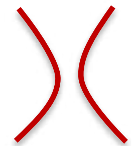
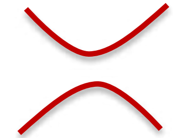
We present a general theory of non-local conserved currents in two-dimensional quantum field theory in the lattice approximation. They reflect quantum group symmetries. Various examples are studied.

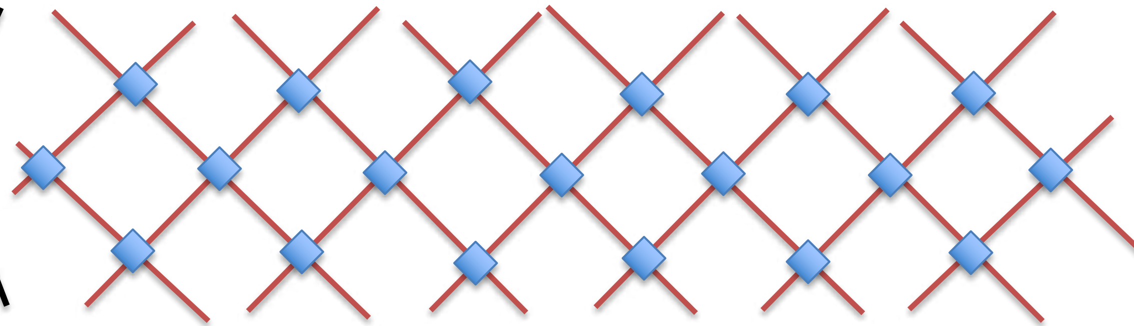
The graphical representation of eqs. (2.7) and (2.8) is then



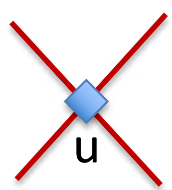
(2.10)

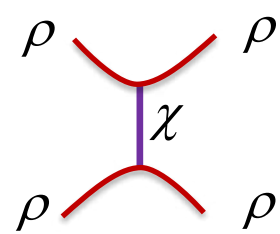
Recall for loops: 

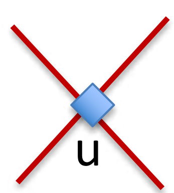
$= v(u)$  $+ \; h(u)$ 

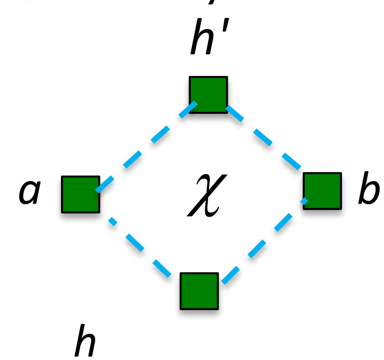
$Z = \text{eval} \Big($  $\Big)$

$$= \sum_{\mathcal{F}} \text{eval}[\mathcal{F}] \prod_{\text{v}} A_{\chi_{\text{v}}}(u)$$

In general, 

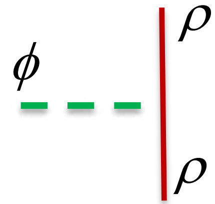
$= \sum_{\chi} A_{\chi}(u)$ 

In local models: 

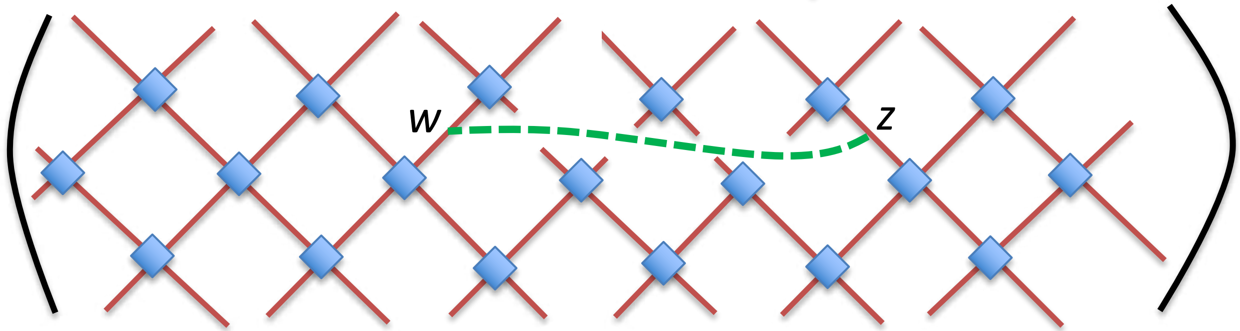
$= \sum_{\chi} A_{\chi}(u)$ 

Defining the currents

Choose an object $\phi \in \rho \otimes \rho$ so that there is a vertex



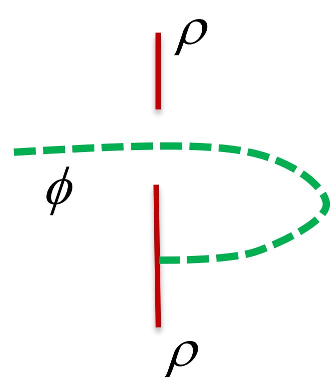
$$\left\langle \overline{J}(w) J(z) \right\rangle = \frac{1}{Z} \text{eval}$$



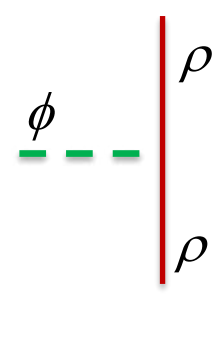
i.e. terminate a topological defect!

Current is non-local: need braiding for string to go over intervening edges.

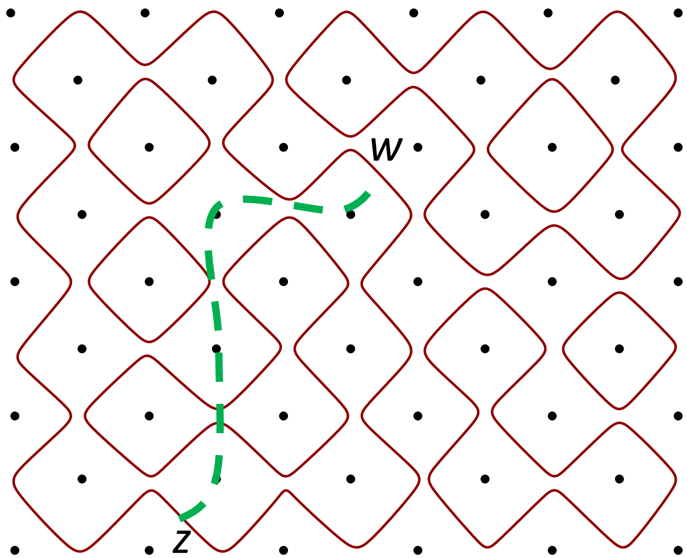
Independent of path except for

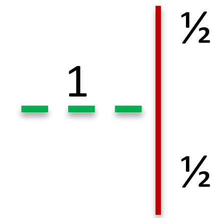


$$= \Omega_\phi^{-1}$$

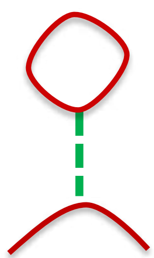


For loops/Temperley-Lieb/Potts/Jones, loops are labeled with ½, currents with 1.



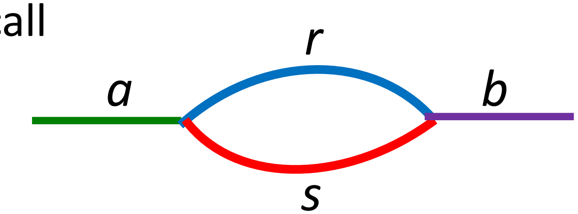


The weight is zero unless the string connects two points on the same loop.



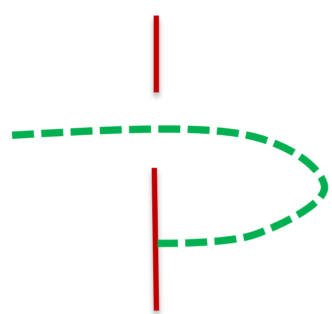
= 0

recall

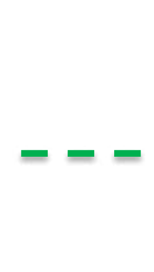


$\propto \delta_{ab}$

But don't forget twist!



$= \Omega_1^{-1}$



Discrete “Holomorphicity”

- Smirnov defined non-local operators in a few lattice models that are discrete ``holomorphic”: they obey half the lattice Cauchy-Riemann equations.

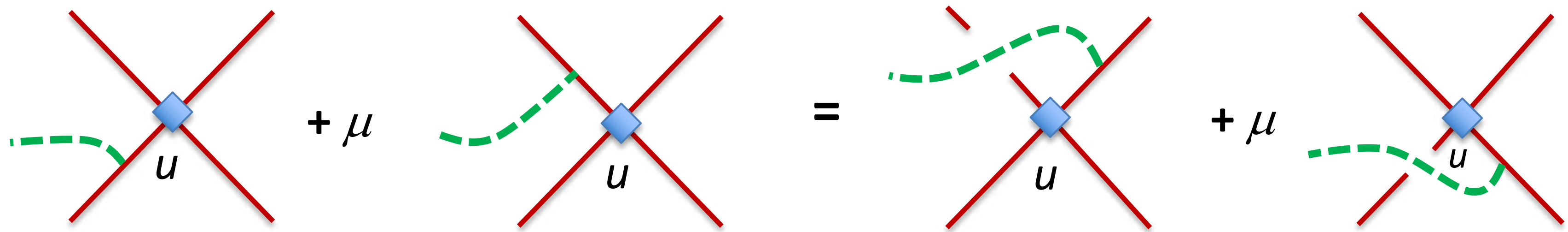
- Examples are fermion operator in the Ising model (which obeys all the lattice C-R equations), or parafermions in the 3-state Potts model. Cardy and collaborators found many more in geometric lattice models.

- Cardy et al also had a different philosophy. They did not require a priori that the Boltzmann weights satisfy the YBE. Requiring this conserved current exist then gives a linear condition for the Boltzmann weights. Solving it gives turns out to yield a solution of the full trilinear Yang-Baxter equation. Baxterization!

Discrete ``holomorphicity’’ = vanishing lattice divergence

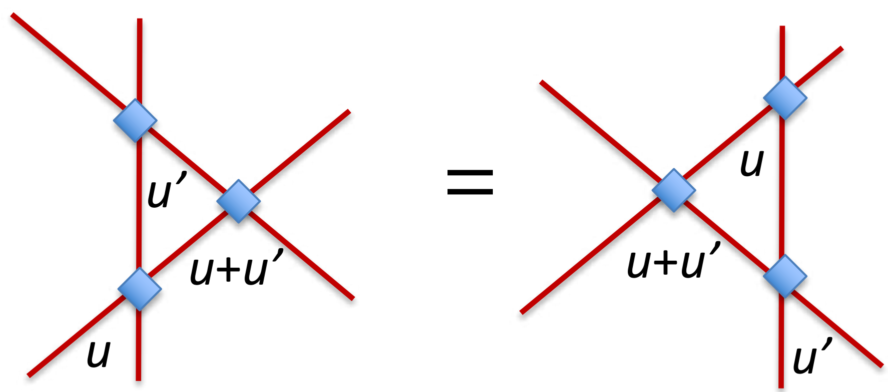
Conserved-current relation in a braided tensor category

All the data save μ and the amplitudes $A_\chi(u)$ are specified by the category



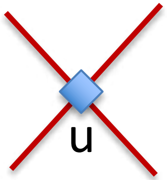
Smirnov et al

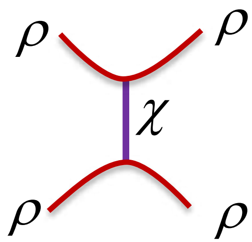
Cardy et al



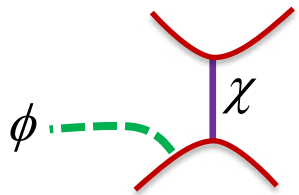
In all known cases, the weights solving the linear equation also solve the YBE!

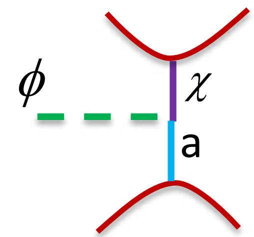
Solving the conserved-current relation

Plug 

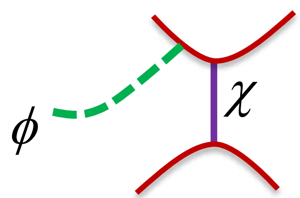
$= \sum_{\chi} A_{\chi}(u)$ 

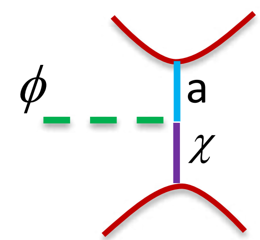
into each of the terms

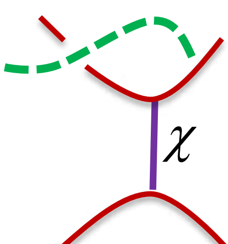


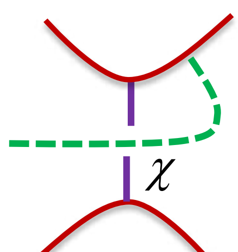
$$= \sum_{a \in \rho \otimes \rho} F_{\rho a} \begin{bmatrix} \phi & \chi \\ \rho & \rho \end{bmatrix}$$


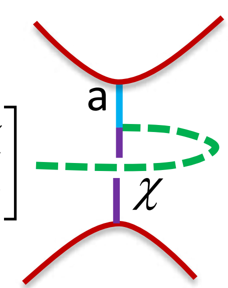
Then manipulate
to a common form:

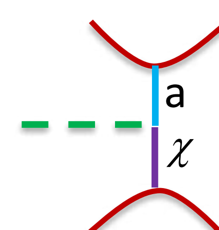


$$= \sum_{a \in \rho \otimes \rho} F_{\rho a} \begin{bmatrix} \chi & \phi \\ \rho & \rho \end{bmatrix}$$


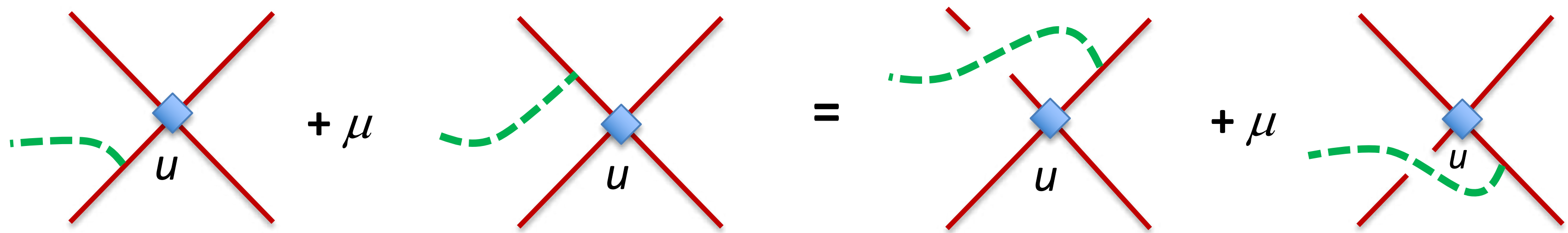


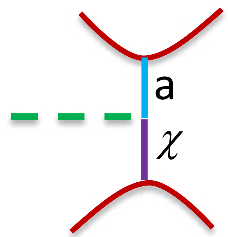
$= \Omega_{\phi}$ 

$= \Omega_{\phi} \sum_{a} F_{\rho a} \begin{bmatrix} \phi & \chi \\ \rho & \rho \end{bmatrix}$ 

$= \sum_{a} \frac{\Omega_{\chi}}{\Omega_{a}} F_{\rho a} \begin{bmatrix} \phi & \chi \\ \rho & \rho \end{bmatrix}$ 

Solving the conserved-current relation



$$0 = \sum_{a,\chi \in \rho\otimes\rho} B_{a\chi}$$ 

$$\Rightarrow \quad B_{a\chi} = 0$$

Find all *F* symbols cancel with the simplifying assumptions made (no arrows, no mutplicities), but analysis has been generalized (T. Osborne, unpublished).

$$\frac{A_\chi(u)}{A_a(u)} = \frac{e^{iu}\Omega_\chi + \Omega_a}{\Omega_\chi + e^{iu}\Omega_a}$$

for all $a, \chi, \phi \in \rho \otimes \rho$

such that $N^{\phi}_{a\chi} \neq 0$

$$\mu \equiv e^{iu}$$

$$\frac{A_\chi(u)}{A_a(u)} = \frac{e^{iu}\Omega_\chi + \Omega_a}{\Omega_\chi + e^{iu}\Omega_a}$$

for all $a, \chi, \phi \in \rho \otimes \rho$

such that $N^{\phi}_{a\chi} \neq 0$

This formula generalizes the quantum-group result of Zhang, Gould and Bracken (using Jimbo's work) to all braided tensor categories and any choice of ϕ .

No need for quantum-group representation theory!

A solution is not guaranteed.

Need to check that ratios are all consistent with $N^{\phi}_{a\chi} \neq 0$, and the Ω_a. Nevertheless, many many solutions exist, sometimes even more than one for a given lattice model.

Smirnov; Riva and Cardy; Rajabpour and Cardy; Ikhlef and Cardy; Ikhlef, Fendley and Cardy; de Gier et al; Batchelor et al; Ikhlef and Weston; Chelkak, Glazman and Smirnov

Future directions

- Would be nice to prove that such weights always give a solution of the YBE.
- Would be nice to have a general criterion for when it works.
- Most resulting models are critical (trigonmetric solutions of the YBE). Elliptic?
- Relax some of the category constraints, e.g. finite number of simple objects.
- For any fusion category, there exists a (more complicated) braided category called the Drinfeld center. Makes plausible very new and different solutions of the YBE, but none are known. Haagerup anyone?