Dynamical spin chains in 4D $\mathcal{N}=2$ SCFTs

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[Review 1912.00870 EP] [2106.08449 Rabe, EP, Zoubos]

Motivation

- Is $\mathcal{N}=4$ SYM the only* integrable theory?
- What happens when we have less supersymmetry?
- Can we do this in an organised way?

The past

- Why do people believe that $\mathcal{N}=2$ theories are not integrable?

[1006.0015 Gadde, EP, Rastelli]

- They do not obey the usual YBE.

- Does this kill integrability? No!

[Review 1912.00870 EP]

Integrable models

- Rational (like XXX based on SU(2))

- Trigonometric (like XXZ based on $SU(2)_q$)

- Elliptic (like XYZ based on $SU(2)_{q,t}$)

- There are also hyper-elliptic examples (chiral potts model)

Elliptic models

- ***Depending on the basis*** we use, elliptic models do not have to obey the standard YBE but a modified, **dynamical YBE**.
  [Felder 1994]

- In the "Baxter basis" (where the **usual YBE** is obeyed) there is **no highest weight state**.

- SCFTs have BPS operators which correspond to the highest weight states. They are naturally <u>**not**</u> **in the "Baxter basis"**.

[Drinfeld 1990]

Quasi-Hopf algebras

✱ There is more than elliptic models and the **dynamical YBE**.

✱ Drinfeld twist: **quasi-Hopf algebras**, **quasi-Hopf YBE**.

$$R_{12}\Phi_{312}R_{13}\Phi^{-1}_{132}R_{23}\Phi_{123} = \Phi_{321}R_{23}\Phi^{-1}_{231}R_{13}\Phi_{213}R_{12}$$

✱ When the Drinfeld twist obeys the so called **shifted cocycle condition**, we get elliptic models and the **dynamical YBE**.

$\mathcal{N}=2$ SCFTs

- Lagrangian $\mathcal{N}=2$ SCFTs are classified. [Bhardwaj,Tachikawa 2013]

- Most of them can be obtain via ***orbifolding*** $\mathcal{N}=4$ SYM and then ***marginally deforming***.

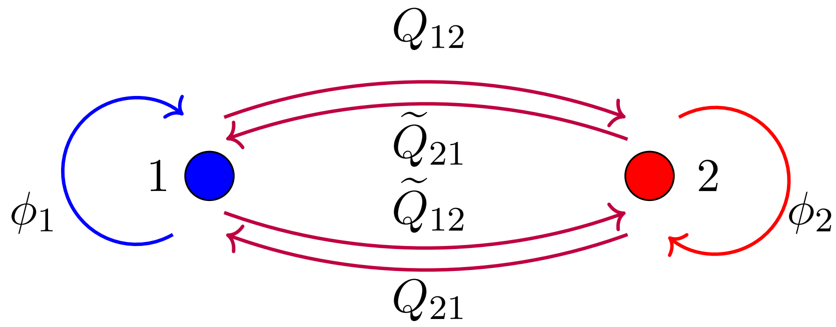
- We know the gravity duals for marginally deformed orbifolds.

- At the ***orbifold point*** (no marginal def.) they are ***integrable***. [Beisert,Roiban 2005]

- We only need to understand how to marginally deform.

Our main example

The Z_2 quiver theory



Z_2 orbifold $\mathcal{N}=4$ SYM and then **marginally deform** away from the **orbifold point** ($g_1=g_2$)

Bifundamental — Adjoint — Adjoint

$$X = \begin{pmatrix} & Q_{12} \\ Q_{21} & \end{pmatrix}, \quad Y = \begin{pmatrix} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \end{pmatrix}, \quad Z = \begin{pmatrix} \phi_1 & \\ & \phi_2 \end{pmatrix}$$

✱ Enough to discover all novel features (dynamical, elliptic …).

✱ When $g_2 \longrightarrow 0$ gives $\mathcal{N}=2$ SCQCD in the Veneziano limit ($N_f=2N_c$).

The Plan of the talk

- The spin chains of $\mathcal{N}=2$ SCFTs are **dynamical.**

- $\mathcal{N}=2$ SCFTs enjoy a **quasi-Hopf symmetry** algebra.

- The **R-matrix** in the *quantum plane limit* and the **twist.**

- The SU(3) scalar sector as a **dynamical 15-vertex model**.

- Explicit study using the coordinate Bethe ansatz.

Dynamical spin chains

XY sector: an alternating spin chain

Every $\mathcal{N}=4$ SYM spin chain state $\Big|XYXYYX\cdots\Big\rangle$

Gives two $\mathcal{N}=2$ spin chain states $\Big|Q_{12}\tilde{Q}_{21}Q_{12}\tilde{Q}_{21}\tilde{Q}_{12}Q_{21}\cdots\Big\rangle$

$\square_1\times\bar{\square}_2\ \square_2\times\bar{\square}_1\ \square_1\times\bar{\square}_2\ \square_2\times\bar{\square}_1\ \square_1\times\bar{\square}_2\ \square_2\times\bar{\square}_1$

Which are Z_2 conjugate $\Big|Q_{21}\tilde{Q}_{12}Q_{21}\tilde{Q}_{12}\tilde{Q}_{21}Q_{12}\cdots\Big\rangle$

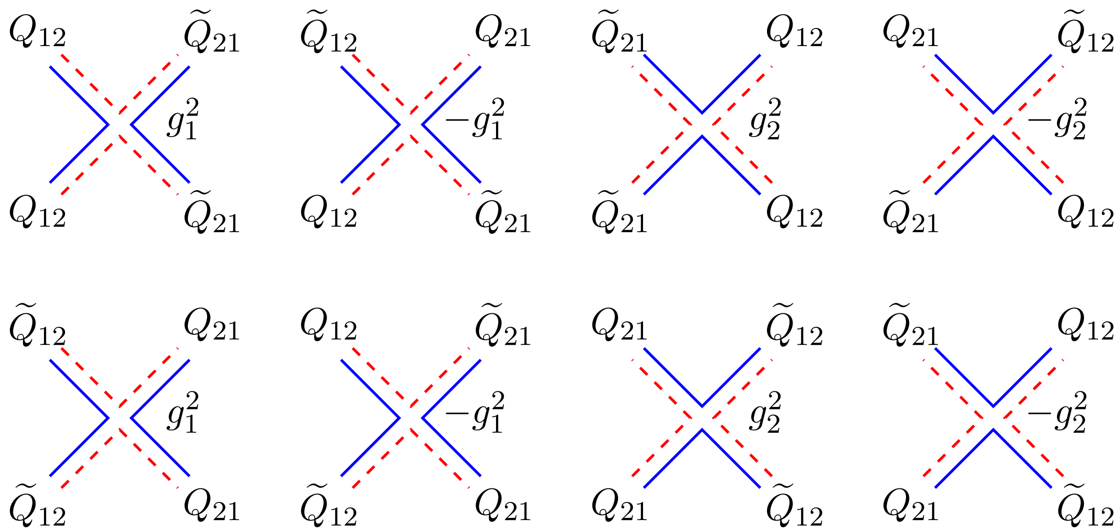
$\square_2\times\bar{\square}_1\ \square_1\times\bar{\square}_2\ \square_2\times\bar{\square}_1\ \square_1\times\bar{\square}_2\ \square_2\times\bar{\square}_1\ \square_1\times\bar{\square}_2$

(k states for a rank k orbifold)

Note that if we **specify** the gauge group of **the first color index** we identify which of the two states we have. This can be done by labelling

$$\Big|XYXYYX\cdots\Big\rangle_{i=1,2}$$

The XY sector Hamiltonian



$$\mathcal{H}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa^{-1} & -\kappa^{-1} & 0 \\ 0 & -\kappa^{-1} & \kappa^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa & -\kappa & 0 \\ 0 & -\kappa & \kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} XX \\ XY \\ YX \\ YY \end{pmatrix}_{i=1,2}$$

$$H_{\ell,\ell+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa^{-1} & -\kappa^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\kappa^{-1} & \kappa^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa & -\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & -\kappa & \kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_{12}Q_{21} \\ Q_{12}\tilde{Q}_{21} \\ \tilde{Q}_{12}Q_{21} \\ \tilde{Q}_{12}\tilde{Q}_{21} \\ Q_{21}Q_{12} \\ Q_{21}\tilde{Q}_{12} \\ \tilde{Q}_{21}Q_{12} \\ \tilde{Q}_{21}\tilde{Q}_{12} \end{pmatrix}$$

Two XXX Hamiltonians with different overall coefficients κ and 1/κ

Dynamical XXX

$$\kappa = \frac{g_2}{g_1}$$

XZ sector: dynamical spin chain

Every $\mathcal{N}=4$ SYM spin chain state $\quad |XZXZZX\cdots\rangle$

Gives two $\mathcal{N}=2$ spin chain states $\quad |Q_{12}\phi_2 Q_{21}\phi_1\phi_1 Q_{12}\cdots\rangle$

$$\square_1\times\overline{\square}_2 \;\; \square_2\times\overline{\square}_2 \;\; \square_2\times\overline{\square}_1 \;\; \square_1\times\overline{\square}_1 \;\; \square_1\times\overline{\square}_1 \;\; \square_1\times\overline{\square}_2$$

Which are Z_2 conjugate $\quad |Q_{21}\phi_1 Q_{12}\phi_2\phi_2 Q_{21}\cdots\rangle$

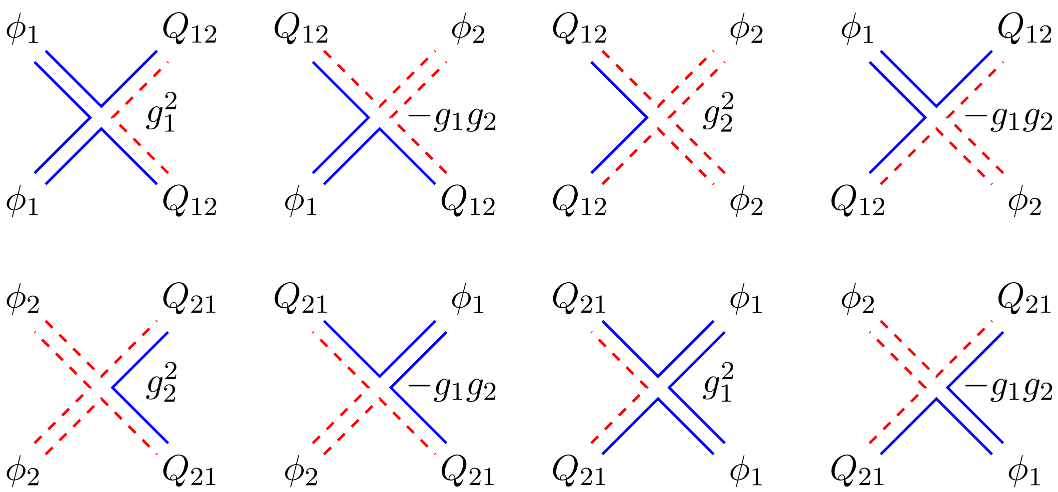
$$\square_2\times\overline{\square}_1 \;\; \square_1\times\overline{\square}_1 \;\; \square_1\times\overline{\square}_2 \;\; \square_2\times\overline{\square}_2 \;\; \square_2\times\overline{\square}_2 \;\; \square_2\times\overline{\square}_1$$

(k states for a rank k orbifold)

We **specify** the gauge group of **the first color index** we identify which of the two states we have. This can be done by labelling

$$|XZXZZX\cdots\rangle_{i=1,2}$$

The XZ sector Hamiltonian



$$\mathcal{H}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa & -1 & 0 \\ 0 & -1 & \kappa^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa^{-1} & -1 & 0 \\ 0 & -1 & \kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} XX \\ XZ \\ ZX \\ ZZ \end{pmatrix}_{i=1,2}$$

$$H_{i,i+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \kappa^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa^{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} Q_{12}Q_{21} \\ Q_{12}\phi_2 \\ \phi_1 Q_{12} \\ \phi_1\phi_1 \\ Q_{21}Q_{12} \\ Q_{21}\phi_1 \\ \phi_2 Q_{21} \\ \phi_2\phi_2 \end{pmatrix}$$

$$\kappa = \frac{g_2}{g_1}$$

Two Temperley-Lieb Hamiltonians with two different deformation parameters κ and 1/κ

Dynamical Temperley-Lieb

Quasi-Hopf symmetry

[Drinfeld 1990]

Quasi-Hopf symmetry

[Roiban2004] [Berenstein,Cherkis2004] [Månsson,Zoubos2008][Dlamini,Zoubos2016&19]

- As for marginal deformations of $\mathcal{N}=4$ SYM.
- $\mathcal{N}=2$ SCFTs enjoy a quasi-Hopf symmetry algebra.
- To discover it look at the F-terms.
- They define a (complex 3D) quantum plane.
- The R-matrix at the *quantum plane limit* (*Braid limit*)

$$\lambda x^b x^a = R^{ab}_{jl} x^j x^l$$

- The superpotential is invariant under the quantum group.

Ex. the Manin quantum plane

$$qxy = yx$$

Can be obtain from an R-matrix:

$$\lambda x^b x^a = R^{ab}_{jl} x^j x^l$$

$$R = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

The quantum plane is invariant under the transformations $x'^i = \mathrm{t}^i_j x^j$.

They obey the algebra $U_q(sl(2))$ which is obtained using the Rtt relations:

$$R^{i\ k}_{\ a\ b} t^a_{\ j} t^b_{\ l} = t^k_{\ b} t^i_{\ a} R^{a\ b}_{\ j\ l}$$

$$\mathrm{t}^1_{\ 2}\mathrm{t}^2_{\ 1} = \mathrm{t}^2_{\ 1}\mathrm{t}^1_{\ 2} \ , \ \mathrm{t}^1_{\ 1}\mathrm{t}^2_{\ 2} - \mathrm{t}^2_{\ 2}\mathrm{t}^1_{\ 1} = (q^{-1} - q)\mathrm{t}^1_{\ 2}\mathrm{t}^2_{\ 1}$$

$$\mathrm{t}^1_{\ 1}\mathrm{t}^1_{\ 2} = q^{-1}\mathrm{t}^1_{\ 2}\mathrm{t}^1_{\ 1} \ , \ \mathrm{t}^1_{\ 1}\mathrm{t}^2_{\ 1} = q^{-1}\mathrm{t}^2_{\ 1}\mathrm{t}^1_{\ 1} \ , \ \mathrm{t}^1_{\ 2}\mathrm{t}^2_{\ 2} = q^{-1}\mathrm{t}^2_{\ 2}\mathrm{t}^1_{\ 2} \ , \ \mathrm{t}^2_{\ 1}\mathrm{t}^2_{\ 2} = q^{-1}\mathrm{t}^2_{\ 2}\mathrm{t}^2_{\ 1}$$

3D quantum planes classified

[Ewen,Ogievetsky1994]

Parameterise using two tensors E_{ijk} and F_{ijk}:

$$E^{\alpha}_{ij}x^i x^j = 0 \qquad u_i u_j F^{ij}_{\alpha} = 0$$

Quantum plane — Quantum co-plane

$$\delta^i_j = \frac{1}{2}E_{jmn}F^{mni}$$

$$E_{ijk}x^i x^j x^k = 0 \qquad u_i u_j u_k F^{ijk} = 0$$

The R-matrix is given by:

$$\hat{R}^{ij}_{\ \ kl} = \delta^i_k \delta^j_l - E_{klm}F^{mij}$$

Using this R-matrix we get back the right quantum plane relations and through the Rtt relations we can write down the quantum algebra (symmetries of the quantum plane)

Used successfully marginally deformed $\mathcal{N}=4$ SYM

[Månsson,Zoubos2008][Dlamini,Zoubos2016&19]

[Roiban2004] [Berenstein,Cherkis2004] [Månsson,Zoubos2008] [Dlamini,Zoubos2016&19]

Leigh–Strassler theory

$$\phi^1\phi^2 = q\phi^2\phi^1 - h(\phi^3)^2$$
$$\phi^2\phi^3 = q\phi^3\phi^2 - h(\phi^1)^2$$
$$\phi^3\phi^1 = q\phi^1\phi^3 - h(\phi^2)^2$$

3D Quantum plane

$$\mathcal{W}_{\mathcal{N}=4} = g\mathrm{Tr}\left\{\Phi^1[\Phi^2,\Phi^3]\right\} = \frac{g}{3}\epsilon_{ijk}\mathrm{Tr}\left\{\Phi^i\Phi^j\Phi^k\right\}$$

↓

$$\mathcal{W}_{LS} + \mathcal{W}_{LS}^\dagger = \frac{1}{3}\mathrm{Tr}\left(E_{ijk}\Phi^i\Phi^j\Phi^k + \overline{\Phi}_i\overline{\Phi}_j\overline{\Phi}_k F^{ijk}\right)$$

The quantum co-plane: hermitian conjugate: $F^{ijk} = \overline{E}_{ijk}$

$$E_{123} = E_{231} = E_{312} = \frac{1}{d}\,,$$
$$E_{321} = E_{213} = E_{132} = -\frac{q}{d}\,,$$
$$E_{111} = E_{222} = E_{333} = \frac{h}{d}\,,$$

$$d^2 = \frac{1+\bar{q}q+\bar{h}h}{2}$$

The Hamiltonian is obtained by: $H^{jk}_{mn} = E_{mna}F^{ajk}$

The R-matrix: $\hat{R}^{ij}_{\ \ kl} = \delta^i_k\delta^j_l - E_{klm}F^{mij}$

$$R = \frac{1}{2d^2}\begin{pmatrix}
1+q\bar{q}-h\bar{h} & 0 & 0 & 0 & 0 & -2\bar{h} & 0 & 2\bar{h}q & 0 \\
0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 & 0 & 0 & 0 & 2h\bar{q} \\
0 & 0 & 2q & 0 & -2h & 0 & q\bar{q}+h\bar{h}-1 & 0 & 0 \\
0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 & 0 & 0 & 0 & -2h \\
0 & 0 & 2\bar{h}q & 0 & 1+q\bar{q}-h\bar{h} & 0 & -2\bar{h} & 0 & 0 \\
2h\bar{q} & 0 & 0 & 0 & 0 & 2\bar{q} & 0 & 1-q\bar{q}+h\bar{h} & 0 \\
0 & 0 & 1-q\bar{q}+h\bar{h} & 0 & 2h\bar{q} & 0 & 2\bar{q} & 0 & 0 \\
-2h & 0 & 0 & 0 & 0 & q\bar{q}+h\bar{h}-1 & 0 & 2q & 0 \\
0 & -2\bar{h} & 0 & 2\bar{h}q & 0 & 0 & 0 & 0 & 1+q\bar{q}-h\bar{h}
\end{pmatrix}$$

The **Lagrangian is invariant** under the transformations $\Phi^i \to \mathrm{t}^i_{\ j}\Phi^j$ which form a quantum version of SU(3) defined by the Rtt relations.

AdS point of view

Gravity dual reason why we have a quantum algebra:

NSNS B-field turned on the C^3 (transverse to the D3)

When there is a B-field the open strings on the D3 branes see a non-commutative geometry.
Open strings see a quantum plane!

[Seiberg,Witten1999]
[Schomerus1999]

✱ For the Leigh–Strassler background [Kulaxizi 2006]

✱ Marginally deformed orbifolds also have a B-field on the orbifolded $C^2 \subset C^3$ (transverse to the D3) allowing us to go away from the orbifold point

$$\frac{1}{g_1^2}+\frac{1}{g_2^2}=\frac{1}{2\pi g_s} \qquad \frac{g_1^2}{g_2^2}=\frac{\beta}{1+\beta} \text{ with } \beta=\int_{S^2} B_{NS}$$

[Gadde, EP, Rastelli 2009]

The Z_2 quiver quantum group

There are two copies (images) of the quantum plane:

$$g_1 Q_{12}\widetilde{Q}_{21} = g_1 \widetilde{Q}_{12} Q_{21} \quad , \quad g_2 Q_{21}\widetilde{Q}_{12} = g_2 \widetilde{Q}_{21} Q_{12}$$

$$\phi_2 Q_{21} = \frac{1}{\kappa} Q_{21}\phi_1 \quad , \quad \phi_1 Q_{12} = \kappa Q_{12}\phi_2$$

$$\phi_2 \widetilde{Q}_{21} = \frac{1}{\kappa} \widetilde{Q}_{21}\phi_1 \quad , \quad \phi_1 \widetilde{Q}_{12} = \kappa \widetilde{Q}_{12}\phi_2$$

(k images for a rank k orbifold)

$$\mathcal{W}_{\mathcal{N}=4} = g \mathrm{Tr}\left\{\Phi^1[\Phi^2, \Phi^3]\right\} = \frac{g}{3}\epsilon_{ijk}\mathrm{Tr}\left\{\Phi^i\Phi^j\Phi^k\right\}$$

$$\mathcal{W} = E^{(1)}_{ijk}\mathrm{Tr}_1\left(X^iX^jX^k\right) + E^{(2)}_{ijk}\mathrm{Tr}_2\left(X^iX^jX^k\right)$$

$$\hat{R}^{ij}_{\;\;kl} = \delta^i_k\delta^j_l - E_{klm}F^{mij}$$

XY sector the R is proportional to the identity

XZ and YZ sector

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2\kappa}{\kappa^2+1} & -\frac{\kappa^2-1}{\kappa^2+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\kappa^2-1}{\kappa^2+1} & \frac{2\kappa}{\kappa^2+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2\kappa}{\kappa^2+1} & \frac{\kappa^2-1}{\kappa^2+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\kappa^2-1}{\kappa^2+1} & \frac{2\kappa}{\kappa^2+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E^{(1)}_{123} = g_1\,,\ E^{(1)}_{231} = g_2\,,\ E^{(1)}_{312} = g_1\,,\ E^{(1)}_{132} = -g_2\,,\ E^{(1)}_{321} = -g_1\,,\ E^{(1)}_{213} = -g_1$$

$$E^{(2)}_{123} = g_2\,,\ E^{(2)}_{231} = g_1\,,\ E^{(2)}_{312} = g_2\,,\ E^{(2)}_{132} = -g_1\,,\ E^{(2)}_{321} = -g_2\,,\ E^{(2)}_{213} = -g_2$$

The Z_2 quiver quantum group

XY sector the R ∝ 1: the SU(2) that rotates X and Y is unbroken (indeed true)

XZ nontrivial R: the SU(2) that rotates X and Z is broken (upgraded to quantum)

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2\kappa}{\kappa^2+1} & -\frac{\kappa^2-1}{\kappa^2+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\kappa^2-1}{\kappa^2+1} & \frac{2\kappa}{\kappa^2+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2\kappa}{\kappa^2+1} & \frac{\kappa^2-1}{\kappa^2+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\kappa^2-1}{\kappa^2+1} & \frac{2\kappa}{\kappa^2+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & k' & 0 & 0 & 0 & 0 & 0 \\ 0 & -k' & k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k & -k' & 0 \\ 0 & 0 & 0 & 0 & 0 & k' & k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The Rtt relations define the quantum group $SU(2)_\kappa$

For the XZ sector there is a twist:

$$R = F_{21} F_{12}^{-1} = (F_{12})^{-2}$$

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & -\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with: $\alpha = \frac{\kappa+1}{\sqrt{2}\sqrt{1+\kappa^2}}$, $\beta = \frac{\kappa-1}{\sqrt{2}\sqrt{1+\kappa^2}}$

A quasi-Hopf symmetry algebra

The Z_2 quiver has extra symmetry

The **superpotential** is **invariant** under a quantum $SU(3)_\kappa$ symmetry, with the appropriate co-product

$$\Delta R^a{}_b = \mathbb{K}_{ab} \otimes R^a{}_b + R^a{}_b \otimes \mathbb{K}_{ba}$$

- The XZ SU(2) (as well as YZ SU(2)) inside the SU(3) are quantum.
- We can do this both in an $\mathcal{N}=2$ and in an $\mathcal{N}=4$ (dynamical) language.

We have the action of the generators of the $SU(3)_\kappa$ on fields in both languages as well as the co-product and we are currently working out the action of the full supergroup $PSU(2,2|4)_\kappa$!

[2111.xxxxx Andriolo,Bertle,EP,Zhang,Zoubos]

Conjecture

- The $\mathcal{N}<4$ theories which can be obtained via ***orbifolding***, ***orientifolding***, … the mother $\mathcal{N}=4$ SYM theory, enjoy a **quantum deformation** of PSU(2,2|4).

- The **naively broken** generators of PSU(2,2|4) → SU(2,2|$\mathcal{N}$) get **upgraded** to *quantum generators*.

… any susy breaking that is due to R-symmetry breaking.

A 15-Vertex model for the SU(3) sector

Vertex models

* The 6-vertex model : XXZ (trigonometric)

* The 8-vertex model : XYZ (elliptic)

* How Baxter solved the 8-vertex model (XYZ): he did a **local change of basis** and made the R-matrix of the 8-vertex to look like the R-matrix of the 6-vertex model (locally).

Elliptic algebras

- The **vertex-type** elliptic algebras: Baxter-Belavin R-matrix **obeys YBE**.

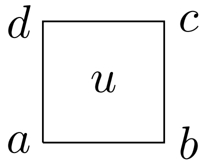
- The **face-type** elliptic algebras: R-matrix of Andrews, Baxter, Forrester. Felder showed that they obey a **dynamical YBE** (DYBE).

- The two algebras are related by a twist. [q-alg/9712029Jimbo,Konno,Odake,Shiraishi]

- The first does not have a highest weight state the second one does (this is why we need the second one)!

Andrews Baxter Forrester

SOS models: statistical (square lattice) models defined by a set of Boltzmann face weights



$$= W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right)$$

u: rapidity a, b, c, d: the heights

Each model comes with a set of rules as to which heights are allowed to be adjacent.

<u>ABF model:</u> ***neighbouring heights can only differ by 1.***

$$W\left(\begin{matrix} a & a+1 \\ a+1 & a+2 \end{matrix}\middle| u\right) = W\left(\begin{matrix} a & a-1 \\ a-1 & a-2 \end{matrix}\middle| u\right) = \frac{\theta_1(2\eta - u)}{\theta_1(2\eta)}$$

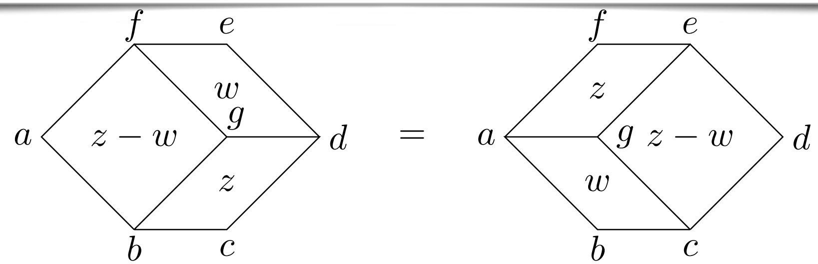
η: Baxter's R_{XYZ}

$$W\left(\begin{matrix} a & a+1 \\ a-1 & a \end{matrix}\middle| u\right) = W\left(\begin{matrix} a & a-1 \\ a+1 & a \end{matrix}\middle| u\right) = \frac{\sqrt{\theta_1(2\eta(a-1)+w_0)\theta_1(2\eta(a+1)+w_0)}}{\theta_1(2\eta a + w_0)} \frac{\theta_1(u)}{\theta_1(2\eta)}$$

$$W\left(\begin{matrix} a & a+1 \\ a+1 & a \end{matrix}\middle| u\right) = \frac{\theta_1(2\eta a + w_0 + u)}{\theta_1(2\eta a + w_0)}\ , \quad W\left(\begin{matrix} a & a-1 \\ a-1 & a \end{matrix}\middle| u\right) = \frac{\theta_1(2\eta a + w_0 - u)}{\theta_1(2\eta a + w_0)}$$

Integrability is captured by the **star-triangle relation**:

$$\sum_g W\left(\begin{matrix} f & g \\ a & b \end{matrix}\middle| z-w\right) W\left(\begin{matrix} g & d \\ b & c \end{matrix}\middle| z\right) W\left(\begin{matrix} f & e \\ g & d \end{matrix}\middle| w\right) = \sum_g W\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| w\right) W\left(\begin{matrix} f & e \\ a & g \end{matrix}\middle| z\right) W\left(\begin{matrix} e & d \\ g & c \end{matrix}\middle| z-w\right)$$



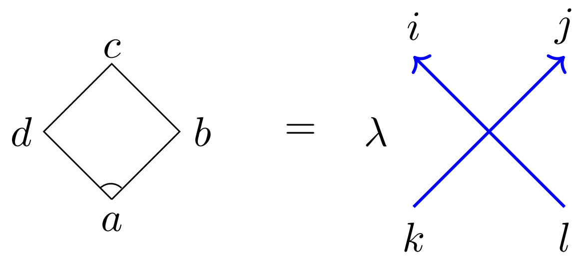
Felder's R-matrix

$$e[1] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; e[-1] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Felder's R-matrix = ABF R-matrix after using the vertex-face map:

$$R(u; -2\eta d) e[c-d] \otimes e[b-c] = \sum_a W \left(\begin{matrix} d & c \\ a & b \end{matrix} \middle| u \right) e[b-a] \otimes e[a-d]$$

$$R(u;\lambda) = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \alpha & \beta_+ & 0 \\ 0 & \beta_- & \alpha & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$$



$$\alpha = \frac{\sqrt{\theta_1(\lambda+2\eta)\theta_1(\lambda-2\eta)}}{\theta_1(-\lambda)} \frac{\theta_1(u)}{\theta_1(2\eta)} \qquad \beta_\pm = \frac{\theta_1(\lambda \pm u)}{\theta_1(\lambda)} \qquad \gamma = \frac{\theta_1(2\eta - u)}{\theta_1(2\eta)}$$

The R-matrix of Felder obeys a dynamical YBE (DYBE)

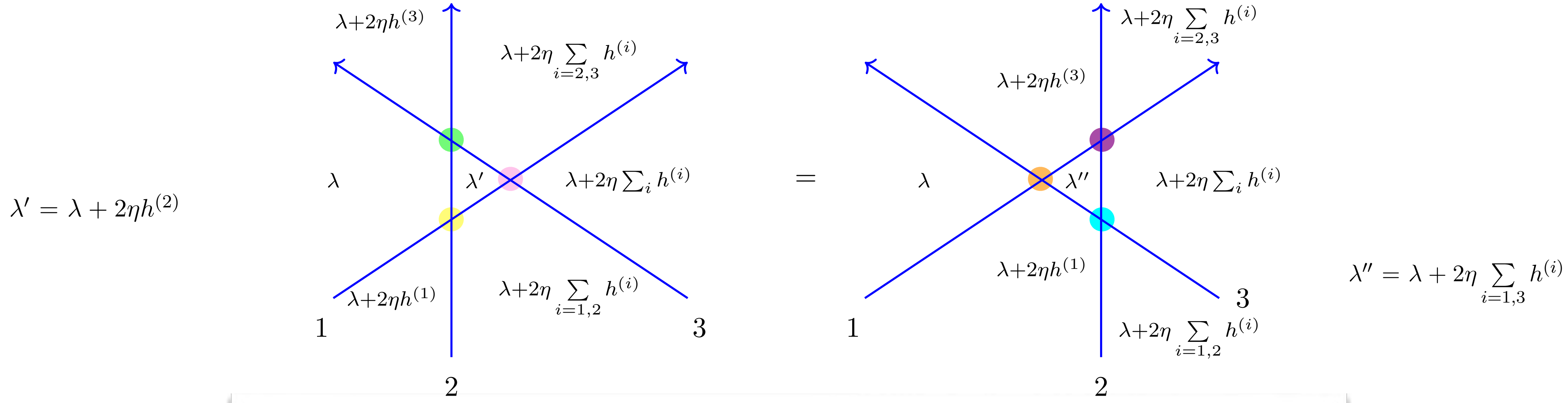
η: Baxter's R_{XYZ}

$$\begin{aligned} &R_{12}(u_1-u_2;\lambda+2\eta h^{(3)}) R_{13}(u_1-u_3;\lambda) R_{23}(u_2-u_3;\lambda+2\eta h^{(1)}) \\ &\qquad = R_{23}(u_2-u_3;\lambda) R_{13}(u_1-u_3;\lambda+2\eta h^{(2)}) R_{12}(u_1-u_2;\lambda) \end{aligned}$$

Important: the **quasi-Hopf YBE** becomes the **DYBE** when the twist satisfies a so-called **shifted cocycle relation.**

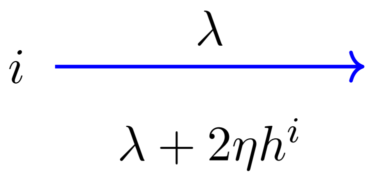
Dynamical YBE

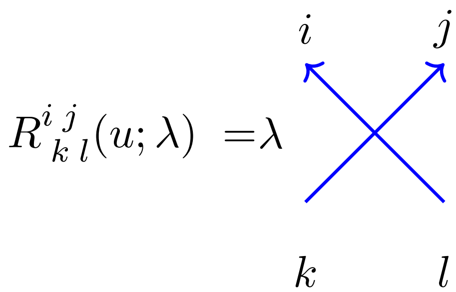
[1701.05562 Yagi]



$$R_{12}(u_1 - u_2; \lambda + 2\eta h^{(3)}) R_{13}(u_1 - u_3; \lambda) R_{23}(u_2 - u_3; \lambda + 2\eta h^{(1)})$$
$$= R_{23}(u_2 - u_3; \lambda) R_{13}(u_1 - u_3; \lambda + 2\eta h^{(2)}) R_{12}(u_1 - u_2; \lambda)$$

The R-matrix of Felder is a function of the ***dynamical parameter λ*** which is ***shifted by 2η*** when we cross an index line





Having the dynamical parameter to always be shifted by 2η is **not good for our purpose!**

Dilute RSOS/CSOS models

[Warnaar, Nienhuis, Seaton, Pearce…]

Having the dynamical parameter to always be shifted by 2η is **not good for our purpose!**

When we cross a Z (field in the adjoint representation) we don't want to shift λ!

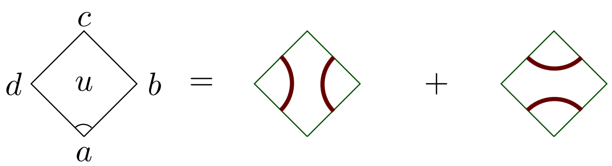
The ***dynamical parameter λ*** will keep track of the ***color group***!

To achieve that we need to study dilute RSOS/CSOS models.

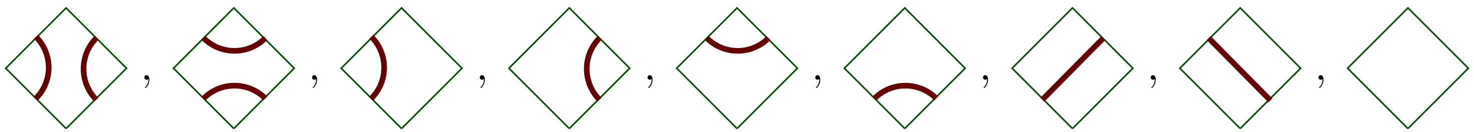
The name dilute comes from the link to loop models.

ABF is a **dense** or **fully packed** face model (***neighbouring heights can differ by 1***)

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) =$$



We need to also have the dilute tiles:



Boltzmann face weights can have ***neighbouring heights differing by 1 or <u>being equal!</u>***

$$W\left(\begin{matrix} d & d\pm1 \\ d\pm1 & d\pm1 \end{matrix}\middle| u\right), W\left(\begin{matrix} d & d \\ d & d\pm1 \end{matrix}\middle| u\right), W\left(\begin{matrix} d & d\pm1 \\ d & d \end{matrix}\middle| u\right), W\left(\begin{matrix} d & d \\ d\pm1 & d \end{matrix}\middle| u\right), W\left(\begin{matrix} d & d \\ d\pm1 & d\pm1 \end{matrix}\middle| u\right), W\left(\begin{matrix} d & d\pm1 \\ d & d\pm1 \end{matrix}\middle| u\right), W\left(\begin{matrix} d & d \\ d & d \end{matrix}\middle| u\right)$$

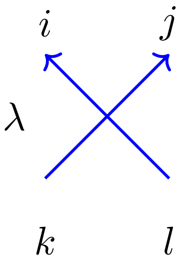
SU(3) sector as a 15-vertex model

Assume there is a vertex model, whose R-matrix $R(u;\kappa) \equiv R(u;\lambda)$

Produces the Hamiltonian $\mathcal{H}(\kappa) \propto \frac{d}{du} R(u;\kappa)|_{u=0}$

The R-matrix is a function of $\kappa(\lambda)$

$R^{i\,j}_{k\,l}(u;\lambda) \; =_\lambda$



Crossing a bifundamental field Q: κ → 1/κ

$$\kappa(\lambda \pm 2\eta) = 1/\kappa(\lambda)$$

In the dynamical spin chain language this corresponds to λ → λ ± 2η

The R matrix must obey

$$R(u;\kappa) \equiv R(u;\lambda) \quad \Leftrightarrow \quad R(u;\kappa^{-1}) \equiv R(u;\lambda \pm 2\eta)$$

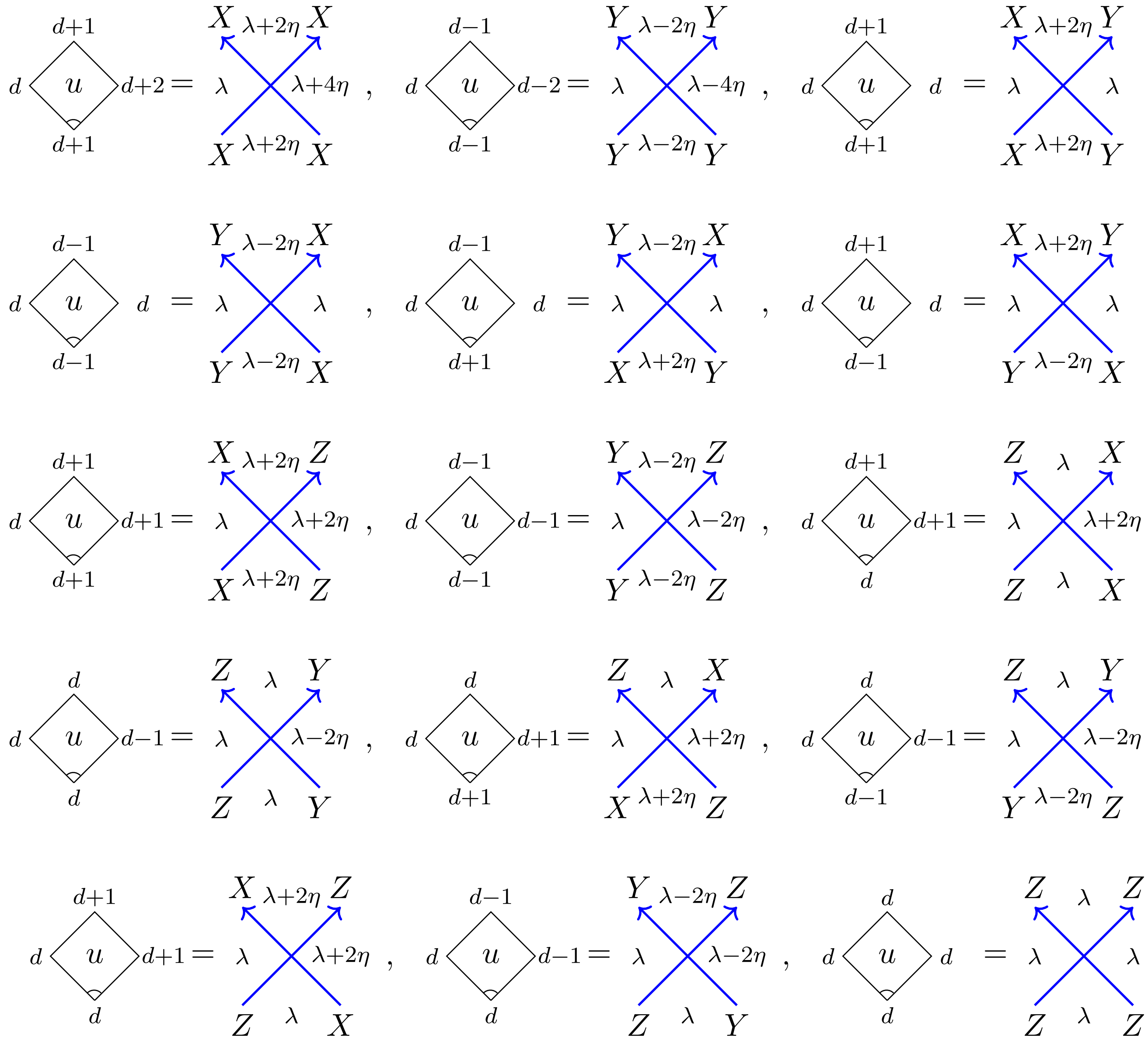
Crossing two bifundamentals: κ → 1/κ → κ we return to the original coupling constant (dynamical parameter λ) thus the periodicity of the model is $\lambda \pm 4\eta \sim \lambda$

$$R(u;\lambda) = R(u;\lambda \pm 4\eta)$$

Crossing an **adjoint** Z field **does not alter** the gauge group and thus the dynamical parameter **λ,** thus the model is **dilute.**

(λ±2kη~λ for a rank k orbifold)

SU(3) sector as a 15-vertex model



15-vertex models for $\mathcal{N}=2$ SCFTs

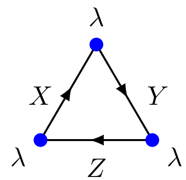
Locally, this 15-vertex model capture the holomorphic SU(3) sector for any $\mathcal{N}=2$ SCFT.

Only difference between different $\mathcal{N}=2$ SCFTs is the topology of the quiver and the "**global periodicity**": how the ***dynamical parameter λ*** get's shifted to capture the possibility for all different ***color groups of the quiver diagram*** and when it comes back to itself.

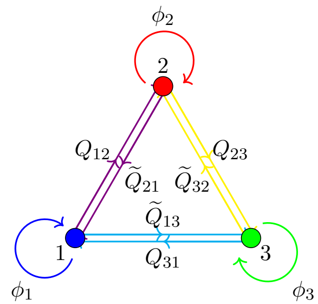
For the Z_2 quiver $\lambda \pm 4\eta \sim \lambda$ For the Z_3 quiver $\lambda \pm 6\eta \sim \lambda$

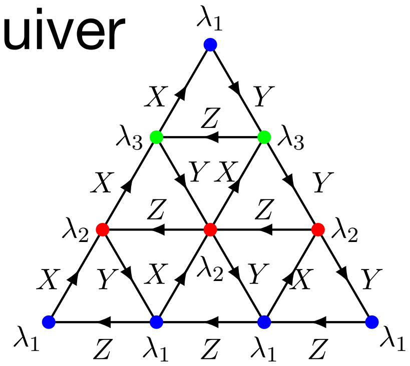
The periodicity is captured by the **adjacency graph.** [Jimbo,Miwa,Okado 1987]

$\mathcal{N}=4$ SYM

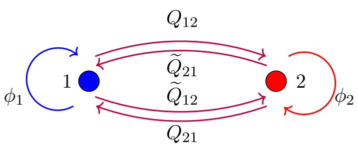


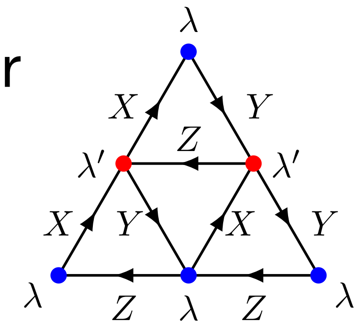
$\mathcal{N}=2$ Z_3 quiver





$\mathcal{N}=2$ Z_2 quiver





Conjectures

* For every $\mathcal{N}=2$ theory the holomorphic SU(3) sector can be captured by a **dynamical 15-vertex** model which is specified by the **adjacency graph**, which is the dual to the **brane-tiling diagram** of the quiver theory.

* Similarly, for a large class of $\mathcal{N}=1$ theories the holomorphic SU(3) sector will be captured by a **dynamical 19-vertex** model which is specified by the **adjacency graph**, which is the dual to the **brane-tiling diagram** of the quiver theory.

A generic $\mathcal{N}=1$ theory can have vertices: XY → ZZ and conjugates which an $\mathcal{N}=2$ cannot due to R-symmetry!

Bethe Ansatz

Explicit Bethe Ansatz

In [1006.0015 Gadde, EP, Rastelli] we studied the XZ sector around the "*phi-vacuum*". The solution looked like **two coupled trigonometric models**, and the **naive YBE was not satisfied**.

Two phi vacua: $|0\rangle \equiv \mathrm{tr}\left(\phi^\ell\right)$ $|\check{0}\rangle \equiv \mathrm{tr}\left(\check{\phi}^\ell\right)$ One for each color group.

Magnons *interpolate* between the *two vacua* $\cdots\phi\,\phi\,\phi\,Q\,\check{\phi}\,\check{\phi}\,\check{\phi}\cdots$ $\cdots\check{\phi}\,\check{\phi}\,\check{\phi}\,\tilde{Q}\,\phi\,\phi\,\phi\cdots$

$$g^2 E(p) = 2(g-\check{g})^2 + 8\,g\,\check{g}\sin^2\left(\frac{p}{2}\right)$$

Two inequivalent two-magnon scatterings $\cdots\phi\,\phi\,\phi\,Q\,\check{\phi}\,\check{\phi}\,\check{\phi}\cdots\check{\phi}\,\check{\phi}\,\check{\phi}\,\tilde{Q}\,\phi\,\phi\,\phi\cdots$ $S = S_{XXZ}(\kappa)$

$\cdots\check{\phi}\,\check{\phi}\,\check{\phi}\,\tilde{Q}\,\phi\,\phi\,\phi\cdots\phi\,\phi\,\phi\,Q\,\check{\phi}\,\check{\phi}\,\check{\phi}\cdots$ $\tilde{S} = S_{XXZ}(\frac{1}{\kappa})$

$$S_{XXZ}(\kappa) = -\frac{1-2\kappa e^{ip_1}+e^{i(p_1+p_2)}}{1-2\kappa e^{ip_2}+e^{i(p_1+p_2)}}$$

YBE not satisfied: $$S\,\tilde{S}\,S \neq \tilde{S}\,S\,\tilde{S}$$

Revisit the explicit 3-body BA in the light of **quasi-Hopf** [Bozkurt, EP, Zoubos]

Explicit Bethe Ansatz

Very different properties manifest when expand around an other vacuum, the "***Q-vacuum***".

$$|Q\rangle \equiv \mathrm{tr}\left(\cdots Q\tilde{Q}Q\tilde{Q}Q\tilde{Q}Q\tilde{Q}Q\tilde{Q}\cdots\right)$$

Even the one-magnon problem reveals novel features!

$$|\phi(p)\rangle \equiv \sum_{\ell} A(p)e^{ip\ell}|\phi_\ell\rangle + \sum_{\ell} B(p)e^{ip\ell}|\check{\phi}_\ell\rangle \,. \qquad r(p) \equiv \frac{B(p)}{A(p)} = \frac{(1-\kappa^2) \pm \sqrt{(1-\kappa^2)^2 + 4\,\kappa^2\cos^2 p}}{2\,\kappa\,\cos p}$$

The dispersion relation is elliptic!

$$E_1(p;\kappa) = \frac{1}{\kappa} + \kappa \pm \frac{1}{\kappa}\sqrt{(1+\kappa^2)^2 - 4\kappa^2\sin^2 p}$$

For two magnons we can find a solution on the center of mass frame using conventional Bethe Ansatz techniques (usual permutations plus nearest neighbour contact terms).

Explicit Bethe Ansatz

It is not possible to find a solution away from the center of mass frame **unless** we use **extra momenta** to parameterise the solution.

$$k_{1,2} = \frac{K}{2} \pm \frac{\pi}{2} \mp \frac{1}{2} \arccos\left(\cos(p_1 - p_2) + \frac{(E_2 - 2(\kappa + 1/\kappa))^2 \cos K}{2 \sin^2 K} \right)$$

$$K = p_1 + p_2$$

This is due to the *elliptic form of the dispersion relation*

$$E_2(p_1, p_2) = 2(\kappa + 1/\kappa) - \sqrt{1 + \kappa^4 + 2\kappa^2 \cos(2p_1)} - \sqrt{1 + \kappa^4 + 2\kappa^2 \cos(2p_1)}$$

the 2 magnon conservation of momentum and energy problem has 2 solutions.

Hinting to that the only correct rapidity is an elliptic one!

$$e^{ip} = i\sqrt{k}\mathrm{sn}(v/\kappa) = i\frac{\theta_1(u)}{\theta_4(u)}$$

$$r(u) = \frac{\sqrt{k}\mathrm{cn}(v/\kappa)}{\mathrm{dn}(v/\kappa)} = \frac{\theta_2(u)}{\theta_3(u)} \qquad \kappa^2 = \left(\frac{\theta_2(0)}{\theta_3(0)}\right)^2$$

$$q = e^{i\pi\tau}\ , \ \text{where} \ \ \tau = i\frac{K'(m)}{K(m)}$$

Interesting eigenvalues under Z_2. Much more to do ….

Conclusions

✱ $\mathcal{N}=2$ SCFTs enjoy a **quantum SU(3)$_\kappa$ symmetry algebra**.

✱ Map the SU(3) scalar sector to a **dynamical 15-vertex model**.

✱ Explicit study with the coordinate Bethe ansatz.

Conclusions

Two Conjectures:

* The $\mathcal{N}<4$ theories which can be obtained via ***orbifolding***, ***orientifolding***, … the mother $\mathcal{N}=4$ SYM theory, enjoy a **quantum deformation** of PSU(2,2|4).

* For $\mathcal{N}<4$ theories the holomorphic SU(3) sector can be captured by a **dynamical 15/19-vertex** model which is specified by the **adjacency graph**, which is the dual to the **brane-tiling diagram** of the quiver theory.

Outlook

* Write down the weights of the 15-vertex models (map to the explicit BA solution) and check if they obey the star-triangle relation. [EP, Zoubos]

* Shifted cocycle condition important for integrability.

* Introduce the rapidity via Baxterization or via the adjoint action.

Outlook

- ✱ Generalise (ellipticise) everything we have for $\mathcal{N}=4$ SYM.

- ✱ Very similar: $\mathcal{N}=1$ SCFTs again starting with orbifolds (big class of theories).

- ✱ Study the gravity dual of marginally deformed orbifolds!

- ✱ "4D Chern-Simons" approach [2005.03064 Costello,Stefański]

- ✱ Generalize [2104.08263 Gaberdiel,Gopakumar]

  The String Dual to Free $\mathcal{N}=2$ SCFTs

Thanks!